

Geometrical approach to hydrodynamics and low-energy excitations of spinor condensatesRyan Barnett,¹ Daniel Podolsky,^{2,3} and Gil Refael¹¹*Department of Physics, California Institute of Technology, MC 114-36, Pasadena, California 91125, USA*²*Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7*³*Department of Physics, Technion, Haifa 32000, Israel*

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In this work, we derive the equations of motion governing the dynamics of spin- F spinor condensates. We pursue a description based on standard physical variables (total density and superfluid velocity), alongside $2F$ “spin nodes:” unit vectors that describe the spin- F state and also exhibit the point-group symmetry of a spinor condensate’s mean-field ground state. In the first part of our analysis, we derive the hydrodynamic equations of motion, which consist of a mass continuity equation, $2F$ Landau-Lifshitz equations for the spin nodes, and a modified Euler equation. In particular, we provide a generalization of the Mermin-Ho relation to spin one and find an analytic solution for the skyrmion texture in the incompressible regime of a spin-half condensate. In the second part, we study the linearized dynamics of spinor condensates. We provide a general method to linearize the equations of motion based on the symmetry of the mean-field ground state using the local stereographic projection of the spin nodes. We also provide a simple construction to extract the collective modes from symmetry considerations alone akin to the analysis of vibrational excitations of polyatomic molecules. Finally, we present a mapping between the spin-wave modes, and the wave functions of electrons in atoms, where the spherical symmetry is degraded by a crystal field. These results demonstrate the beautiful geometrical structure that underlies the dynamics of spinor condensates.

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I. INTRODUCTION

A central theme of contemporary atomic physics experiment is the dynamics of Bose-Einstein condensates and other correlated atomic gases. Of particular interest are mixtures of several species such as Fermi-Bose mixtures^{1–3} and bosonic gases with an internal spin degrees of freedom, *i.e.*, spinor condensates. Spin-one and spin-two spinor condensates have been realized as particular hyperfine states of alkali atoms.^{4–6} In addition the trapping and cooling of ⁵²Cr atoms have led to the realization of a spin-three condensate.⁷ On the theoretical front, since the initial work of Ohmi and Machida,⁸ and Ho⁹ numerous interesting works have followed that discuss ground states, dynamics, and topological excitations of such systems (see, for instance, a review in Ref. 10).

Recent theoretical interests in spinor condensates have focused on topics such as dynamics near the insulating transition,¹¹ metastable decay of currents,¹² spin knots,¹³ and the anomalous Hall effect.¹⁴ One particularly important aspect of spinor condensates is their free dynamics under a time-dependent Hamiltonian, about ground or metastable states. This aspect was the center of several experimental^{15–20} and theoretical^{21–27} studies. These investigations, so far, were mostly confined to the simplest case of spin-one condensates. On the other hand, the wealth and intricacy of spinor condensates increases dramatically with increasing spins. For instance, the phase diagram of spin-two and spin-three condensates consists of four and ten possible mean-field phases, respectively.^{8,9,28–31} A feature which makes these systems even more interesting is that the ground states exhibit a high degree of symmetry in its spin state which is isomorphic to lattice point groups.^{32–34} In this paper we seek to utilize this symmetry in the study of the free dynamics of spinor condensates.

Recently, it was shown within mean-field theory that the ground states of spinor condensates exhibit a high degree of symmetry. This symmetry is opaque in the standard spinor description of the condensate. On the other hand, the symmetry is transparent in the so-called reciprocal-state representation. Here, one uses the fact that the mean-field ground state of a spin- F condensate can be described by $2F$ coherent-spin states orthogonal to it. Each one of these so-called reciprocal states is fully spin polarized, pointing along some direction on the unit sphere. Since there are $2F$ such reciprocal states, the ground state is uniquely described (up to an overall phase) in terms of the $2F$ points on the unit sphere.³² For typical spinor-condensate Hamiltonians, these points (or antipodes which we denote spin nodes, see Sec. III), form highly symmetric configurations. For instance, an $F=2$ condensate has a cyclic phase, where the spin nodes are arranged in a tetrahedron, as well as a square phase.

The spin-node description of the ground states of spinor condensates provides an intuitive geometrical description of the state of the condensate. In addition, it provides a parametrization which readily exhibits the hidden point-group symmetries of the state. Despite its appeal, however, this parametrization has not been used to describe the dynamics of spinor condensates. Our goal in this paper is to provide a complete description of the hydrodynamics of spinor condensates in terms of such spin nodes.

The first aspect we consider is the continuum hydrodynamics of spinor condensates (Secs. III–VI). Our description is hydrodynamic in the sense that it focuses on the low-energy dynamics of the system associated with locally conserved quantities or with the slow elastic deformation of spontaneously broken degrees of freedom. Here we derive such a description using the density, superfluid velocity, and the spin nodes (the $2F$ vectors on the unit sphere) as our basic degrees of freedom. In addition to the Euler equations,

which describe mass, momentum, and energy conservation, we obtain $2F$ Landau-Lifshitz equations for the dynamics of the spin nodes. Furthermore, our derivation gives a natural generalization of the Mermin-Ho relation which connects the vorticity in a ferromagnetic spinor condensate with the Pontryagin density of the order parameter.

The treatment of the spin degrees of freedom in this part is exact and it accounts for the full geometrical structure of the hydrodynamics of spinor condensates. But the precision of the hydrodynamic description here comes at a price: this formalism becomes increasingly complex as the spin F grows, and, for large F , the analysis of its exact form becomes impractical. Nevertheless, the equations derived here even for large F become quite useful in their linearized form.

In the remainder of the paper we show how linearizing the equations of motion about mean-field solutions elucidate the low-energy properties of spinor condensates with arbitrary spin in a powerful and elegant way. We consider the general $2F$ spin-node description of the low-lying spin-wave excitations near the mean-field ground state (Secs. VII and VIII). We derive the linearized equations of motion for the spin-node locations, which allows us to extract the small oscillation spectrum from symmetry alone, in a fashion resembling the vibrational-mode calculation for polyatomic molecules,^{35,36} though slightly more complicated. Using this method we are able to give simple expressions for the vibration eigenmodes and energy spectrum. In addition, we derive a correspondence between the low-lying excitations of the spinor condensates and atomic orbitals subject to rotational symmetry due to crystal fields, which reflect the symmetry of the spinor-condensate ground state.

The paper is organized as follows. Section II provides general background on spinor condensates and reviews recent progress on the hydrodynamic description of ferromagnetic condensates.³⁷ In Sec. III we present the spin-node representation of spinor-condensate degrees of freedom and derive several useful identities within this formalism. In Sec. IV we proceed to obtain hydrodynamic equations for the spin-half condensate, using the spin-node formalism, and find an analytic solution for a skyrmion configuration. In Secs. V and VI, we derive the general hydrodynamic equations of motion for the spin-one condensate, and then for an arbitrary spin- F condensate, which includes a generalization of the Mermin-Ho relation.³⁸ In the second part of the paper we concentrate on small deviations from the mean-field ground states. In Sec. VII we derive the linearized equations of motion about the mean-field configuration in terms of the spin-node formalism. Finally, in Sec. VIII we demonstrate how to use symmetry arguments to compute the spin-wave excitations and give a prescription to obtain closed-form expressions for both eigenmodes and eigenenergies of the low-lying spin waves.

II. BACKGROUND

A. Hydrodynamics of spinless BECs

For a single-component Bose-Einstein condensate (BEC), it is natural to expect a simple hydrodynamic description in terms of density and flow velocity. We take the time-

dependent Gross-Pitaevskii equation (GPE) as our starting point

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + g\rho\psi, \quad (1)$$

where $\psi = \sqrt{\rho}e^{i\theta}$ is the macroscopic wave function and $\rho = |\psi|^2$ is the density (here and after for notational simplicity we will use scaled units). This equation can be recast into the form of local momentum and mass conservation laws, with the superfluid velocity $\mathbf{v} = \nabla\theta$, one obtains³⁹

$$\partial_t\rho = -\nabla\cdot(\rho\mathbf{v}); \quad D_t\mathbf{v} = -\nabla\left(g\rho - \frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}}\right), \quad (2)$$

where $D_t = \partial_t + \mathbf{v}\cdot\nabla$ is the material derivative. The first of these is the mass continuity equation while the second is the Euler equation for a fluid, where a quantum pressure term appears.

B. The hydrodynamics of ferromagnetic BECs and the Mermin-Ho relation

In a series of recent experiments, the quench dynamics of a ferromagnetic spin-one condensate was explored.^{17–20} These experiments motivated Lamacraft to develop a hydrodynamic framework for the ferromagnetic BEC in terms of the superfluid velocity and the director of its ferromagnetic order \mathbf{n} .³⁷ This description is particularly illuminating when considering the instabilities of the system. This problem was also theoretically considered in Refs. 21–27.

The GP Lagrangian density describing such a ferromagnetic spinor condensate is given by

$$\mathcal{L} = i\psi_a^*\partial_t\psi_a - \frac{1}{2}\nabla\psi_a^*\cdot\nabla\psi_a - \frac{1}{2}g\rho^2 - \frac{1}{2}c_2\rho^2m^2, \quad (3)$$

where $a = -F, \dots, F$ is summed over all F_z eigenstates, and

$$\rho = \sum_a \psi_a^*\psi_a \quad \mathbf{m} = \frac{1}{\rho} \sum_{ab} \psi_a^*\mathbf{F}_{ab}\psi_b, \quad (4)$$

where \mathbf{F}_{ab} is the spin- F matrix. Lamacraft's approach assumed an incompressible liquid with a wavefunction restricted to the ferromagnetic phase (assuming large g and c_2)

$$\psi_a = e^{i\theta}\Phi_a(\mathbf{n}), \quad (5)$$

where $\Phi_a(\mathbf{n})$ is the highest eigenstate of $\mathbf{n}\cdot\mathbf{F}$. Note that for the ferromagnetic state we have $\mathbf{m} = \mathbf{n}$ (while for the polar state $|\mathbf{m}| = 0$). A substitution of this wavefunction into Eq. (3) yielded the following set of hydrodynamic equations³⁷

$$\nabla\cdot\mathbf{v} = 0,$$

$$D_t\mathbf{n} = \frac{1}{2}\mathbf{n}\times\nabla^2\mathbf{n}. \quad (6)$$

Once the density is eliminated, we notice that the spin dynamics are given by a Landau-Lifshitz equation with the material derivative $D_t = \partial_t + \mathbf{v}\cdot\nabla$. In addition, the vorticity is related to the Pontryagin density by

$$\nabla \times \mathbf{v} = \frac{F}{2} \varepsilon_{\alpha\beta\gamma} n_\alpha (\nabla n_\beta \times \nabla n_\gamma). \quad (7)$$

This identity is widely known as the Mermin-Ho relation.^{9,38} Among other things, such a relation has important consequences for the topological defects in ferromagnetic condensates.^{9,40–42} Such hydrodynamic equations were also derived in Ref. 43 to describe magnetic properties of quantum-hall systems.

Making use of this simple description, Lamacraft showed that the helical configuration of the ferromagnetic condensate¹⁹ is unstable. In general, it is clear that such a geometric description simplifies, at least conceptually, the analysis of spinor-condensate dynamics.

C. General magnetic ground state of spinor condensates and the reciprocal-state representation

A general spin- F spinor condensate is described by a macroscopic wave function with $2F+1$ complex components ψ_a ($a=-F, \dots, F$). When quantum fluctuations are unimportant, the condensate dynamics is described by the time-dependent Gross-Pitaevskii equation

$$i\partial_t \psi_a = -\frac{1}{2} \nabla^2 \psi_a + \frac{\partial V_{\text{int}}}{\partial \psi_a^*}, \quad (8)$$

where V_{int} is the spin-dependent interaction energy. The interaction energy is given by the set of parameters g_S with $S=0, 2, \dots, 2F$ describing the two-particle interaction strength in the S total angular-momentum channel

$$V_{\text{int}} = \frac{1}{2} \sum_{S,m} g_S \psi_a^* \psi_b^* \langle ab | Sm \rangle \langle Sm | a' b' \rangle \psi_{a'} \psi_{b'}. \quad (9)$$

In the above, $\langle ab | Sm \rangle$ are Clebsch-Gordan coefficients. Note that this expression can also be written as the expectation value of an operator

$$V_{\text{int}} = \frac{1}{2} \langle \psi | {}_2 \langle \psi | \mathcal{V}_{\text{int}} | \psi \rangle_2 | \psi \rangle_1, \quad (10)$$

where

$$\mathcal{V}_{\text{int}} = \sum_{S,m} g_S |Sm\rangle \langle Sm| = \sum_S g_S \mathcal{P}_S. \quad (11)$$

In this expression, \mathcal{P}_S projects into the total-spin S scattering channel.

The classical (mean-field) ground states occur for uniform condensates which minimize Eq. (9) for fixed density ρ . This minimization was carried out for $F=1$,^{8,9} $F=2$,²⁸ and $F=3$ (Refs. 30 and 31) yielding a multitude of magnetic phases, which minimize V_{int} in different regions of $\{g_S\}$ parameter space, only one of which (for every F) is ferromagnetic.

Indeed, quite generally, a spin- F spinor condensate may exhibit several flavors of paramagnetic rather than ferromagnetic behavior in its ground state. For example, a spin-one condensate may exhibit the so-called nematic phase, where $\psi_1 = \psi_{-1} = 0$, and $\psi_0 = 1$. The expectation value of the magnetization for such a state is clearly zero along any direction, $\langle \mathbf{F} \cdot \mathbf{n} \rangle = 0$. But in the absence of a ferromagnetic director, \mathbf{n} ,

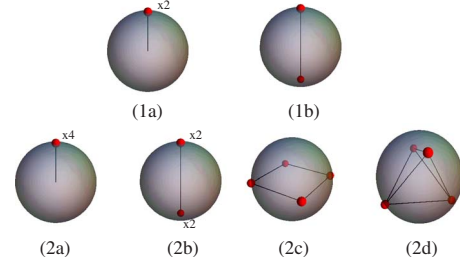


FIG. 1. (Color online) Possible phases for the spin-one and spin-two condensates. The red dots on the unit sphere correspond to the unit vectors \mathbf{n}_i (reciprocal spinors) defined in Eq. (12). Spin-one condensates have ferromagnetic (1a) and nematic phases (1b) while spin-two condensates have ferromagnetic (2a), uniaxial nematic (2b), square biaxial nematic (2c), and a tetrahedral (2d) phases.

can we still describe a spinor condensate's magnetic state geometrically?

Such a geometrical method was put forward in Ref. 32 based on the use of spin-coherent states. A spin-coherent state $|\Phi_{\mathbf{n}}\rangle$ is the eigenvector of the operator $\mathbf{F} \cdot \mathbf{n}$ with the largest eigenvalue. The method of Refs. 32 relies on finding the set of $2F$ spin-coherent states, $\{|\Phi_{\mathbf{n}_i}\rangle\}_{i=1}^{2F}$, which are orthogonal to the ground state of a uniform condensate

$$\langle \Phi_{\mathbf{n}_i} | \psi_{GS} \rangle = 0. \quad (12)$$

The $2F$ states $|\Phi_{\mathbf{n}_i}\rangle$ provide (up to an overall phase) a unique description of the magnetic spin state of the condensate at each point in space. Such reciprocal spinors give a natural generalization of the ferromagnetic director to the case of paramagnetic condensates. Instead of the geometrically opaque $2F+1$ complex numbers ψ_a , it allows a description of the magnetic state in terms of $2F$ unit vectors, \mathbf{n}_i , or points on the unit sphere.

In addition to its geometrical transparency, such a description also reveals the highly symmetric nature of the mean-field ground states. All the paramagnetic phases found so far correspond to a spin-node configuration which is invariant under point symmetry-group operations and sometimes under a larger symmetry. The nematic phase of the $F=1$ condensate, for instance, is described by two antipodal spin nodes. $F=2$ condensates can exhibit a nematic phase as well but also a phase in which the spin nodes are the vertices of a square and a phase with the spin nodes at the vertices of a tetrahedron. Such phases are illustrated in Fig. 1.⁴⁴

The reciprocal-spinor description was so far only utilized to discuss equilibrium properties of spinor condensates. The remarkable geometrical properties and hidden symmetries of the mean-field ground state, however, provide ample motivation for employing the spin nodes to obtain a complete description of the dynamics of spinor condensates. In the following sections we will develop the tools necessary for such a description and use them to derive both a hydrodynamic description as well as small oscillation dynamics near mean-field ground states.

III. SPIN-NODE DESCRIPTION OF SPIN- F MAGNETIC STATES

The reciprocal-spinor states so far define the spinor condensate's state only implicitly through Eq. (12). In order to

be able to use these variables directly, we must invert the relationship, expressing the spinor-condensate Lagrangian directly in terms of these variables. In order to find this direct representation, we first separate the wave function into a piece corresponding to the overall density and phase and a piece describing the local spin state. We write

$$\psi_a = \psi \chi_a, \quad (13)$$

where χ_a is a normalized spin- F spinor

$$\sum_a \chi_a^* \chi_a = \langle \chi | \chi \rangle = 1 \quad (14)$$

and the superfluid density is

$$\rho = |\psi|^2. \quad (15)$$

A. Symmetrized spin-node representation

As discussed above, a spin- F spinor $|\chi\rangle$ can be described by $2F$ reciprocal states. On the other hand, such a state can also be described by a fully symmetrized collection of $2F$ spin-half states. Each spin-half state can be parametrized in terms of two coordinates $\Omega = (\theta, \phi)$ on the unit sphere,

$$|\Omega\rangle = \cos\left(\frac{\theta}{2}\right) e^{i\phi/2} |\uparrow\rangle + \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} |\downarrow\rangle. \quad (16)$$

In this representation

$$|\chi\rangle = \frac{1}{\mathcal{N}} \sum_{\{\sigma\}} \left(\prod_{i=1}^{2F} \otimes |\Omega_{\sigma_i}\rangle \right) = \frac{\sqrt{(2F)!}}{\mathcal{N}} |\mathbf{\Omega}\rangle, \quad (17)$$

where \mathcal{N} is a normalization constant and the sum over σ runs over the $(2F)!$ permutations of the $2F$ labels for the spin-half parts.⁴⁵ In Eq. (17) we also defined $|\mathbf{\Omega}\rangle$ as the (unnormalized) sum over permutations of the tensor product.

The properties of the above formulation are most easily understood using the Schwinger-boson construction⁴⁶ (for review, see Ref. 47). Schwinger bosons provide an easy way to construct the Hilbert space of a spin- F spinor state. We define two Schwinger-boson creation operators: $\hat{a}^\dagger, \hat{b}^\dagger$. An \hat{a}^\dagger boson adds $1/2$ to both the total spin, and to F_z , whereas a \hat{b}^\dagger boson adds $1/2$ to the total spin, but lowers F_z by half. In this notation

$$\begin{aligned} F_{\text{tot}} &= \frac{1}{2} (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}); \\ F_x &= \frac{1}{2} (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}); \\ F_y &= \frac{1}{2i} (\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}); \\ F_z &= \frac{1}{2} (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}). \end{aligned} \quad (18)$$

A spin-half spinor is written as

$$|\Omega\rangle = u |\uparrow\rangle + v |\downarrow\rangle = (u \hat{a}^\dagger + v \hat{b}^\dagger) |0\rangle$$

with $|0\rangle$ the Schwinger-boson vacuum. Here, u and v can be written in terms of the coordinates on the unit sphere as $u = \cos(\theta/2) e^{i\phi/2}$ and $v = \sin(\theta/2) e^{-i\phi/2}$.

A symmetrized tensor product of $2F$ spins within the SB formalism is simply written as

$$|\mathbf{\Omega}\rangle = |\Omega_1 \cdots \Omega_{2F}\rangle = \prod_{i=1}^{2F} (u_i \hat{a}_i^\dagger + v_i \hat{b}_i^\dagger) |0\rangle \quad (19)$$

with u_i and v_i parametrized in terms of θ_i, ϕ_i as shown above. We refer to this collection of the $2F$ spin-half states which construct $|\mathbf{\Omega}\rangle$ as spin nodes.

If we wish to calculate wavefunction overlaps using the Schwinger Boson formalism, we can use Wick's theorem to obtain

$$\begin{aligned} \langle \mathbf{\Omega}^{(a)} | \mathbf{\Omega}^{(b)} \rangle &= \langle 0 | \prod_{i=1}^{2F} (u_i^{(a)*} \hat{a}_i + v_i^{(a)*} \hat{b}_i) \prod_{j=1}^{2F} (u_j^{(b)} \hat{a}_j^\dagger + v_j^{(b)} \hat{b}_j^\dagger) |0\rangle \\ &= \sum_{\{\sigma\}} \prod_{i=1}^{2F} \langle 0 | (u_i^{(a)*} \hat{a}_i + v_i^{(a)*} \hat{b}_i) (u_{\sigma_i}^{(b)} \hat{a}_{\sigma_i}^\dagger + v_{\sigma_i}^{(b)} \hat{b}_{\sigma_i}^\dagger) |0\rangle \\ &= \sum_{\{\sigma\}} \prod_{i=1}^{2F} \langle \Omega_i^{(a)} | \Omega_{\sigma_i}^{(b)} \rangle, \end{aligned} \quad (20)$$

where σ is a permutation of the $2F$ indices that mark the spin-half parts. This result could have also been obtained directly from Eq. (17). Nevertheless, we find it instructive to demonstrate the simple Schwinger-boson construction to obtain the symmetrized states.

B. Connection between spin nodes and reciprocal spinors

Since the symmetrized spin-node representation can be used to express any spin state directly, it makes a grossly overcomplete basis. Nevertheless, its usefulness arises since it perfectly reflects the spin-nodes formalism of the spinor-condensates ground states.^{32,48} In the following we will introduce the necessary new notation for the spin-node formalism; we summarize the new notation in Appendix A.

It is simple to see that a spin-coherent state can be written in terms of Schwinger-boson states as

$$|(\Omega)^{2F}\rangle = (u \hat{a}^\dagger + v \hat{b}^\dagger)^{2F} |0\rangle = |\Omega \cdots \Omega\rangle. \quad (21)$$

Thus a coherent state can be thought of as $2F$ spin nodes pointing in the same direction. (For a summary of the notation see Appendix A)

As described in Sec. II C, a reciprocal spinor is a coherent state $|(\Omega_r)^{2F}\rangle = |\Omega_r \cdots \Omega_r\rangle$ orthogonal to a given spinor $|\mathbf{\Omega}\rangle$. Using the construction in terms of symmetrized spin nodes, we can write an equation to determine the reciprocal spinors for a particular state $|\mathbf{\Omega}\rangle = |\Omega_1 \cdots \Omega_{2F}\rangle$

$$\langle (\Omega_r)^{2F} | \Omega \rangle = (2F)! \prod_{i=1}^{2F} \langle \Omega_r | \Omega_i \rangle = 0. \quad (22)$$

This equation has $2F$ solutions, each corresponding to a different term in the product vanishing. That is, the i th solution of Eq. (22) is

$$|(\Omega_r)^{2F}\rangle = |(\Omega_i^t)^{2F}\rangle = |\Omega_i^t \cdots \Omega_i^t\rangle, \quad (23)$$

where

$$|\Omega_i^t\rangle = (-v_i^* \hat{a}^\dagger + u_i^* \hat{b}^\dagger) |0\rangle \quad (24)$$

is the time-reversed spinor of $|\Omega_i\rangle$ with $\theta_i^t = \pi - \theta_i$ and $\phi_i^t = \phi_i + \pi$. Here we use the fact that a spin-half spinor is orthogonal to its time-reversed counterpart.

Finally, we note that the direction of the spin-half spinor $|\Omega_i^t\rangle$ is opposite to that of $|\Omega_i\rangle$. Explicitly, we have

$$\langle \Omega_i^t | \mathbf{F} | \Omega_i^t \rangle = - \langle \Omega_i | \mathbf{F} | \Omega_i \rangle. \quad (25)$$

One therefore sees that the set of reciprocal spinors is nothing more than the spin nodes pointing in the opposite direction. For example, given the spin-node representation of a particular spinor: $\{\mathbf{n}_i\}_{i=1}^{2F}$, the reciprocal-spinor representation is simply $\{-\mathbf{n}_i\}_{i=1}^{2F}$. From this point on we reserve the symbol \mathbf{n}_i for spin nodes (and not reciprocal spinors). Furthermore, our analysis will be in terms of spin nodes alone.

C. Time derivatives of spinors and spin nodes

As stated above, our goal in this paper is to extract the dynamics in terms of individual spin nodes $|\Omega_i\rangle$. In order to do so, we must be able to isolate the dynamics of each spin node within $|\Omega\rangle$. Consider the time derivative of $|\Omega\rangle$ which will appear in the GPE. We can express it as a sum of terms in which the time derivative operates on individual spin nodes

$$\partial_t |\Omega\rangle = |\partial_t \Omega_1 \Omega_2 \dots\rangle + |\Omega_1 \partial_t \Omega_2 \dots\rangle + \dots \quad (26)$$

The trick that allows us to isolate individual spin nodes consists of taking the inner product of $\partial_t |\Omega\rangle$ with the i th reciprocal state of $|\Omega\rangle$, which is $|(\Omega_i^t)^{2F}\rangle$. All terms which do not involve a time derivative of $|\Omega_i\rangle$ identically vanish and we are left with the single term

$$\langle (\Omega_i^t)^{2F} | \partial_t |\Omega\rangle = (2F)! \langle \Omega_i^t | \partial_t \Omega_i \rangle \prod_{\substack{j=1 \\ j \neq i}}^{2F} \langle \Omega_i^t | \Omega_j \rangle. \quad (27)$$

We will make extensive use of this method for isolating the dynamics of individual spin nodes in the following sections.

D. Geometrical parametrization of the spin-half components: Moving from $|\Omega_i\rangle$ to \mathbf{n}_i

All results above were concerned with breaking a spin- F spinor into its $2F$ spin-half parts, $|\Omega_i\rangle$, and with the correspondence between these spin-half parts and the reciprocal coherent states. We would like, however, to understand the dynamics of spinor condensates not only in terms of the

spinors, $|\Omega_i\rangle$, but also in terms of the unit vectors that describe them, \mathbf{n}_i , where $|\Omega_i\rangle$ is highest value eigenvector of $\mathbf{F} \cdot \mathbf{n}_i$.

The first step in finding the equations of motion in terms of the spin directors \mathbf{n}_i , is to establish an orthonormal triad $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{n})$ that parameterizes the space on S^2 in the vicinity of \mathbf{n}_i . In the following we will only consider a single spin-half part, and therefore we drop the index i .

The first element of the triad is \mathbf{n} itself

$$\mathbf{n} = 2\langle \Omega | \mathbf{F} | \Omega \rangle, \quad (28)$$

where \mathbf{F} is the spin operator, acting in the spin-half Hilbert space. To complete the triad, we again use the time-reversed spin-half ket, $|\Omega^t\rangle$, where $\langle \Omega^t | \Omega \rangle = 0$. With this, we can construct states pointing in the “ x ” and “ y ” directions with respect to \mathbf{n} as

$$|\Omega_x\rangle = \frac{1}{\sqrt{2}}(|\Omega\rangle + |\Omega^t\rangle), \quad (29)$$

$$|\Omega_y\rangle = \frac{1}{\sqrt{2}}(|\Omega\rangle + i|\Omega^t\rangle). \quad (30)$$

These states allow us to complete the orthonormal triad by defining

$$\mathbf{e}_x = 2\langle \Omega_x | \mathbf{F} | \Omega_x \rangle, \quad \mathbf{e}_y = 2\langle \Omega_y | \mathbf{F} | \Omega_y \rangle. \quad (31)$$

From these we can construct

$$\mathbf{e}_\pm = \mathbf{e}_x \pm i\mathbf{e}_y. \quad (32)$$

It is useful to note that $\mathbf{F} \cdot \mathbf{e}_\pm$ act as raising and lowering operators. That is,

$$\mathbf{F} \cdot \mathbf{e}_+ |\Omega\rangle = 0; \quad \mathbf{F} \cdot \mathbf{e}_+ |\Omega^t\rangle = |\Omega\rangle \quad (33)$$

with similar relations holding for lowering operators.

Note that there is an ambiguity in such coordinate systems since \mathbf{e}_x and \mathbf{e}_y can together be rotated about \mathbf{n} which corresponds to the gauge choice for the spinors. That is, the gauge of a spinor can be changed by $|\Omega\rangle \rightarrow e^{i\lambda} |\Omega\rangle$ without changing its direction \mathbf{n} . In general, quantities which are gauge invariant cannot depend on the parameterization of the spin and will only involve the unit vectors \mathbf{n}_i . We will adhere to convention of the spin-half state introduced in Eq. (16). Here the gauge is fixed by requiring that the product of the spin-up and spin-down components of the spinor is real. In this gauge choice it is easy to see that

$$\mathbf{e}_x = \hat{\theta} \quad \mathbf{e}_y = \hat{\phi}, \quad (34)$$

where $\hat{\theta}$ and $\hat{\phi}$ are unit vectors from the spherical coordinate system.

To complete the discussion, we make two observations that will simplify the following analysis. First, we express \mathbf{F} in the basis of our triad as

$$\begin{aligned} \mathbf{F} &= (\mathbf{F} \cdot \mathbf{n})\mathbf{n} + (\mathbf{F} \cdot \mathbf{e}_x)\mathbf{e}_x + (\mathbf{F} \cdot \mathbf{e}_y)\mathbf{e}_y \\ &= (\mathbf{F} \cdot \mathbf{n})\mathbf{n} + \frac{1}{2}(\mathbf{F} \cdot \mathbf{e}_+)\mathbf{e}_- + \frac{1}{2}(\mathbf{F} \cdot \mathbf{e}_-)\mathbf{e}_+. \end{aligned} \quad (35)$$

In addition we note that we can use the spin operator \mathbf{F} as a

projection onto $|\Omega\rangle$ and its time-reversed partner $|\Omega^t\rangle$ by

$$|\Omega\rangle\langle\Omega| = \frac{1}{2} + \mathbf{n} \cdot \mathbf{F} \quad (36)$$

and

$$|\Omega^t\rangle\langle\Omega^t| = \frac{1}{2} - \mathbf{n} \cdot \mathbf{F}. \quad (37)$$

E. Derivatives of spin-half spinors in terms of the triad $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{n})$

The relations derived and recalled in the previous sections allow us to also write derivatives of spinors in terms of vector quantities and their derivatives. The terms that we will encounter arise from terms such as the isolated time derivatives in Eq. (27). Let us now find this decomposition in terms of the triad $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{n})$ and its differential forms.

Our goal is thus to find

$$a_\alpha = i\langle\Omega|\partial_\alpha\Omega\rangle \quad \text{and} \quad \langle\Omega^t|\partial_\alpha\Omega^t\rangle \quad (38)$$

in terms of $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{n})$ and their derivatives. We define a_α in this form for reasons that will become clear later. The first object in Eq. (38) can be found by considering the quantity

$$\partial_\alpha(\langle\Omega^t|\mathbf{e}_- \cdot \mathbf{F}|\Omega\rangle) = \partial_\alpha 1 = 0. \quad (39)$$

Allowing the derivative to operate on the bra, the ket, and the vector \mathbf{e}_- , we find

$$\langle\partial_\alpha\Omega^t|\Omega^t\rangle + \langle\Omega|\partial_\alpha\Omega\rangle = -\frac{1}{2}\mathbf{e}_+ \cdot \partial_\alpha\mathbf{e}_- \quad (40)$$

On the left-hand side we used the facts that $\mathbf{F} \cdot \mathbf{e}_-|\Omega\rangle = |\Omega^t\rangle$ and $\langle\Omega^t|\mathbf{F} \cdot \mathbf{e}_- = \langle\Omega|$. On the right-hand side we used the fact that $\langle\Omega^t|\mathbf{F}|\Omega\rangle = \frac{1}{2}\mathbf{e}_+$ which can be verified from Eq. (35). It is easy to verify that

$$\begin{aligned} \langle\partial_\alpha\Omega^t|\Omega^t\rangle &= \langle\Omega|\partial_\alpha\Omega\rangle, \\ \mathbf{e}_x \cdot \partial_\alpha\mathbf{e}_y &= -\mathbf{e}_y \cdot \partial_\alpha\mathbf{e}_x \end{aligned}$$

from which we find

$$a_\alpha = \frac{1}{2}\mathbf{e}_y \cdot \partial_\alpha\mathbf{e}_x \quad (41)$$

which is the desired result.

To obtain $\langle\Omega^t|\partial_\alpha\Omega\rangle$ we use a similar trick. Starting with

$$0 = \langle\Omega^t|\Omega^t\rangle\langle\Omega^t|\Omega\rangle = \langle\Omega^t|\frac{1}{2} - \mathbf{n} \cdot \mathbf{F}|\Omega\rangle \quad (42)$$

we find

$$\partial_\alpha\left(\langle\Omega^t|\frac{1}{2} - \mathbf{n} \cdot \mathbf{F}|\Omega\rangle\right) = 0. \quad (43)$$

As before, allowing the differentiation to act on the bra, the ket, and \mathbf{n} results in

$$\langle\Omega^t|\partial_\alpha\Omega\rangle = \langle\Omega^t|(\partial_\alpha\mathbf{n}) \cdot \mathbf{F}|\Omega\rangle, \quad (44)$$

where the term with $\langle\partial_\alpha\Omega^t|$ vanishes since $(\frac{1}{2} - \mathbf{n} \cdot \mathbf{F})|\Omega\rangle = 0$. Now, using again the decomposition in Eq. (35), we readily find

$$\langle\Omega^t|\partial_\alpha\Omega\rangle = \frac{1}{2}\mathbf{e}_+ \cdot \partial_\alpha\mathbf{n}. \quad (45)$$

This concludes all the tools we will need for our analysis below. We have found how to directly write a spin- F spinor in terms of its spin nodes, and extract terms having to do with individual spin nodes out of sums arising, e.g., from differentiation. Furthermore we translated the spin-half representation of $|\Omega_i\rangle$ to a set of $2F$ triad bases $(\mathbf{e}_{ix}, \mathbf{e}_{iy}, \mathbf{n}_i)$ which will allow us to parametrize the spin state geometrically. Appendix A summarizes the various notation introduced throughout this section.

IV. HYDRODYNAMICS OF SPIN-HALF CONDENSATES

One of our main goals is to write the exact (mean-field) equations of motion for a spinor condensate in terms of the spin nodes and the superfluid velocity and density. In this section we achieve this goal for spin-half condensates. The equations of motion can be trivially generalized to general spin- F condensates restricted to the ferromagnetic state, when the spinor $|\chi\rangle$ is restricted to be a coherent spin state. In this case the equations of motion for the condensate reduce to those we find below.

A. Gross-Pitaevskii Lagrangian

In this section we consider the Gross-Pitaevskii Lagrangian. We begin by writing the Lagrangian in a revealing form, using the representation of the bosonic field which separates the spinor order parameter into a product of a density piece and a spin piece, $\psi_a = \psi\chi_a$. The GP Lagrangian is then

$$\begin{aligned} \mathcal{L} &= i\psi_a^* \partial_t \psi_a - \frac{1}{2} \nabla \psi_a^* \cdot \nabla \psi_a - V_{\text{int}}, \\ &= i\psi^* \partial_t \psi + \rho a_t - \frac{1}{2} |(-i\nabla - \mathbf{a})\psi|^2 \\ &\quad - \frac{1}{2} \rho Y - V_{\text{int}}, \end{aligned} \quad (46)$$

where V_{int} is the spin-related interaction and $\rho = |\psi|^2$ with ψ a complex field. Eq. (46) defines the spin vector potential

$$a_t \equiv i\langle\chi|\partial_t\chi\rangle; \quad \mathbf{a} \equiv i\langle\chi|\nabla\chi\rangle \quad (47)$$

and the quantity

$$Y \equiv \langle\partial_\alpha\chi|\partial_\alpha\chi\rangle - \langle\chi|\partial_\alpha\chi\rangle\langle\partial_\alpha\chi|\chi\rangle. \quad (48)$$

An interesting observation is that the quantity Y for a general spin $F=N/2$ can be identified with the CP^N model from quantum-field theory.⁴⁹ Notice that there is a $U(1)$ gauge freedom in the density-spin decomposition

$$\psi \rightarrow e^{i\lambda} \psi, |\chi\rangle \rightarrow e^{-i\lambda} |\chi\rangle,$$

where α is implicitly summed over. The quantity Y , however, is gauge independent. We make a gauge choice when we write the normalized $|\chi\rangle$ as in Eq. (17) with $|\Omega\rangle$ written as Eq. (19). This $U(1)$ gauge freedom is also reflected in an ambiguity in the choice of the triad arising from the spin-half parts of $|\chi\rangle \propto |\Omega\rangle$ since for each spin part \mathbf{e}_x and \mathbf{e}_y can together be rotated about \mathbf{n} . The choice of a particular triad is set by the gauge choice. In general, quantities which are gauge invariant cannot depend on the parametrization of the spin and will only involve the unit vectors \mathbf{n}_i . The vector potential can be related to the superfluid velocity by

$$\mathbf{v} = \frac{1}{\rho 2i} (\psi_a^* \nabla \psi_a - \psi_a \nabla \psi_a^*) = \nabla \theta - \mathbf{a}, \quad (49)$$

where θ is the argument of ψ . So far we have not used the fact that the spin is $F=1/2$.

B. Geometric representation of hydrodynamic quantities

Now that we know the quantities of interest in the spinor description of the GP Lagrangian, we can translate them to the hydrodynamic variables of density and magnetization direction. The most important quantity appearing above is the vector potential as defined in Eq. (47). Following the discussion in Sec. III D we see that for a spin-half condensate the vector potential is

$$a_\alpha = i \langle \Omega | \partial_\alpha \Omega \rangle = \frac{1}{2} \mathbf{e}_y \cdot \partial_\alpha \mathbf{e}_x. \quad (50)$$

The analogy between \mathbf{a} and the vector potential appearing in the Maxwell equations compels us to consider the antisymmetric field tensor $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$. Through a series of manipulations this can be written purely in terms of \mathbf{n}

$$\begin{aligned} f_{\alpha\beta} &= \frac{1}{2} \partial_\alpha \mathbf{e}_y \cdot \partial_\beta \mathbf{e}_x - \frac{1}{2} \partial_\beta \mathbf{e}_y \cdot \partial_\alpha \mathbf{e}_x = \frac{1}{2} (\partial_\alpha \mathbf{e}_y \cdot \mathbf{n}) (\partial_\beta \mathbf{e}_x \cdot \mathbf{n}) \\ &\quad - \frac{1}{2} (\partial_\beta \mathbf{e}_y \cdot \mathbf{n}) (\partial_\alpha \mathbf{e}_x \cdot \mathbf{n}) = \frac{1}{2} (\mathbf{e}_y \cdot \partial_\alpha \mathbf{n}) (\mathbf{e}_x \cdot \partial_\beta \mathbf{n}) \\ &\quad - \frac{1}{2} (\mathbf{e}_y \cdot \partial_\beta \mathbf{n}) (\mathbf{e}_x \cdot \partial_\alpha \mathbf{n}) \\ &= \frac{1}{2} (\mathbf{e}_y \times \mathbf{e}_x) \cdot (\partial_\alpha \mathbf{n} \times \partial_\beta \mathbf{n}) = -\frac{1}{2} \mathbf{n} \cdot (\partial_\alpha \mathbf{n} \times \partial_\beta \mathbf{n}). \end{aligned} \quad (51)$$

Note that in the above we have repeatedly used the fact that $\mathbf{v} \cdot \partial \mathbf{v} = 0$ for any unit vector \mathbf{v} . The result is the Pontryagin topological density, which is the object of the celebrated Mermin-Ho relation for spin-half spinors.^{38,50}

The only remaining term is the gauge-invariant quantity Y , defined in Eq. (48). For a spin-half state, we find

$$\begin{aligned} Y &= \langle \partial_\alpha \Omega | \partial_\alpha \Omega \rangle - \langle \partial_\alpha \Omega | \Omega \rangle \langle \Omega | \partial_\alpha \Omega \rangle = \langle \partial_\alpha \Omega | \Omega' \rangle \langle \Omega' | \partial_\alpha \Omega \rangle \\ &= \frac{1}{4} (\mathbf{e}_- \cdot \partial_\alpha \mathbf{n}) (\mathbf{e}_+ \cdot \partial_\alpha \mathbf{n}) = \frac{1}{4} (\partial_\alpha \mathbf{n}) \cdot (\partial_\alpha \mathbf{n}). \end{aligned} \quad (52)$$

Thus the Y term signifies the stiffness of the superfluid with

respect to magnetic gradients [as opposed to simply $U(1)$ phase gradients]. Also, since we identified Y with the Lagrangian density of a CP^1 model, we now reaffirm its equivalence with the nonlinear sigma model.⁴⁹

C. Equations of motion for spin-half condensates

Now that we clarified how the hydrodynamic variables arise in the GP Lagrangian density, we are ready to approach the GP equations of motion. In terms of the original variables, the time-dependent GPE for a spin-half condensate is

$$i \partial_t \psi_a = -\frac{1}{2} \nabla^2 \psi_a + g \rho \psi_a, \quad (53)$$

where we note that the interaction energy for this case is

$$V_{\text{int}} = \frac{1}{2} g \rho^2. \quad (54)$$

Following the substitution $\psi_a = \psi \chi_a$, with χ_a the entries of the spin-half spinor $|\chi\rangle$, and contraction with $\langle \chi |$ we find

$$i \partial_t \psi + \psi a_t = \frac{1}{2} (-i \nabla - \mathbf{a})^2 \psi + \frac{1}{2} Y \psi + g \rho \psi,$$

where $a_\alpha = (a_t, \mathbf{a})$ is the vector potential introduced previously. Substituting $\psi = f e^{i\theta}$ and multiplying both sides of the equation by $e^{-i\theta}$ gives

$$\begin{aligned} i \partial_t f - \partial_t \theta f + f a_t &= \frac{1}{2} (-\nabla^2 f - i f \nabla \cdot \mathbf{v} - 2i \mathbf{v} \cdot \nabla f + f v^2) + \frac{1}{2} Y f \\ &\quad + g \rho f, \end{aligned} \quad (55)$$

where $\mathbf{v} = \nabla \theta - \mathbf{a}$. The imaginary part of this gives

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) \quad (56)$$

which is a mass conservation equation. On the other hand, taking the real part gives

$$\partial_t \theta + \frac{1}{2} v^2 - a_t = \frac{1}{2} \frac{\nabla^2 f}{f} - \frac{1}{2} Y - g \rho. \quad (57)$$

We take the gradient of both sides of this equation [using the identity $\nabla(v^2) = 2(\mathbf{v} \cdot \nabla) \mathbf{v} + 2\mathbf{v} \times (\nabla \times \mathbf{v})$] to get

$$D_t \mathbf{v} = \mathbf{e} + (\mathbf{v} \times \mathbf{b}) - \nabla \left(g \rho + \frac{1}{2} Y - \frac{\nabla^2 \sqrt{\rho}}{2\sqrt{\rho}} \right). \quad (58)$$

In this we have defined the ‘‘electric’’ and ‘‘magnetic’’ fields \mathbf{e} and \mathbf{b} in the usual way from the vector potential. That is, $\mathbf{e}_\alpha = f_{\alpha t}$ and $\mathbf{b}_\alpha = (\nabla \times \mathbf{a})_\alpha = \frac{1}{2} \epsilon_{\alpha\beta\gamma} f_{\beta\gamma}$, with α, β and γ indicating space directions, and the f tensor defined below. Also, note that we have used the material derivative $D_t = \partial_t + \mathbf{v} \cdot \nabla$. The ‘‘electromagnetic force’’ appearing in the right-hand side of the Euler equation is a new feature that is not present in single-component condensates. This new type of quantum pressure arises from nonuniform spin textures in spinor condensates.

Now we move on to find the equations describing the spin dynamics. To do this, we contract the GPE with the time-reversed spinor $\langle \chi_t |$. This gives

$$i\langle\chi_t|D_t\chi\rangle = -\left[\frac{1}{2}\left(2i\mathbf{a}\cdot\langle\chi_t|\nabla\chi\rangle + 2\frac{\nabla f}{f}\cdot\langle\chi_t|\nabla\chi\rangle + \langle\chi_t|\nabla^2\chi\rangle\right)\right]. \quad (59)$$

Using the spin identities developed in Sec. III the following relations can be derived $\langle\chi_t|\nabla^2\chi\rangle = \frac{1}{2}\partial_\alpha(\mathbf{e}_+\cdot\partial_\alpha\mathbf{n})$ and $\langle\partial_\alpha\chi|\chi\rangle\langle\chi_t|\partial_\alpha\chi\rangle = -\frac{1}{4}\partial_\alpha\mathbf{e}_+\cdot\partial_\alpha\mathbf{n}$. In terms of vectors, the above equation is then

$$\frac{i}{2}\mathbf{e}_+\cdot D_t\mathbf{n} = -\frac{1}{2}\left(\frac{1}{2}\mathbf{e}_+\cdot\nabla^2\mathbf{n} + \frac{\partial_\alpha f}{f}\mathbf{e}_+\cdot\partial_\alpha\mathbf{n}\right) \quad (60)$$

which can be rewritten as

$$\rho D_t\mathbf{n} = \frac{1}{2}[\mathbf{n}\times\partial_\alpha(\rho\partial_\alpha\mathbf{n})] \quad (61)$$

which is a Landau-Lifshitz equation.

Thus, collecting everything, we can write down a complete set of equations describing the dynamics of a spin-half condensate

$$-\partial_t\rho = \nabla\cdot(\rho\mathbf{v}),$$

$$\nabla\times\mathbf{v} = -\mathbf{b},$$

$$D_t\mathbf{v} = \boldsymbol{\epsilon} + (\mathbf{v}\times\mathbf{b}) - \nabla\left(g\rho + \frac{1}{8}(\partial_\alpha\mathbf{n})^2 - \frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}}\right),$$

$$\rho D_t\mathbf{n} = \frac{1}{2}[\mathbf{n}\times\partial_\alpha(\rho\partial_\alpha\mathbf{n})],$$

where $\boldsymbol{\epsilon}$ and \mathbf{b} are related to the spin direction through the field tensor

$$f_{\alpha\beta} = \begin{pmatrix} 0 & -\mathbf{e}_x & -\mathbf{e}_y & -\mathbf{e}_z \\ \mathbf{e}_x & 0 & \mathbf{b}_z & -\mathbf{b}_y \\ \mathbf{e}_y & -\mathbf{b}_z & 0 & \mathbf{b}_x \\ \mathbf{e}_z & \mathbf{b}_y & -\mathbf{b}_x & 0 \end{pmatrix} = -\frac{1}{2}\mathbf{n}\cdot(\partial_\alpha\mathbf{n}\times\partial_\beta\mathbf{n}). \quad (62)$$

It is interesting to compare these results with those obtained in the incompressible regime Eqs. (6) which were first derived in Ref. 37. The above equations of motion show that lifting the incompressibility constraint leads to the interesting appearance of a Lorentz force in the Euler equation where the effective electric and magnetic fields are given by the Mermin-Ho relation. In addition, the superfluid density now enters the Landau-Lifshitz equation.

D. Application: Skyrmion texture

As an example of the efficiency of the above hydrodynamic equations of motion, let us consider a specific calculation: skyrmion textures in ferromagnetic condensates. For a standard $U(1)$ vortex, the superfluid velocity close to the vortex core diverges as $1/r$. For a scalar condensate, this causes the superfluid density to be depleted in a small region of order of the coherence length around the core. This can be

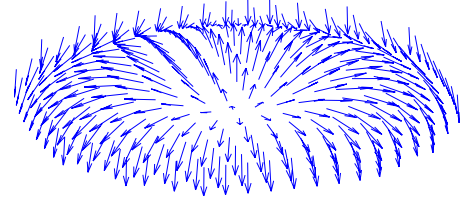


FIG. 2. (Color online) A skyrmion configuration corresponding to Eq. (66).

energetically costly if the condensate is near the incompressible regime. On the other hand, this situation can be circumvented for a spinor condensate. Consider for example, a two-component (spin-half) condensate $(\psi_\uparrow, \psi_\downarrow)$ and take the \downarrow component to have a $U(1)$ vortex. Then around the vortex core, the density of ψ_\downarrow can be transferred to the vortex free ψ_\uparrow keeping the total density across the vortex core finite. This is known as the skyrmion configuration which has been argued to be the relevant topological defect for ferromagnetic condensates.^{9,41,51}

Let us now derive the analytic time-independent solution of the equations of motion in the incompressible regime having the skyrmion texture shown in Fig. 2. To this end, we take the incompressible limit³⁷ of the equations of motion for the spin-half condensate obtained in Sec. IV C. Neglecting z dependence, these are

$$\nabla\cdot\mathbf{v} = 0, \quad (63)$$

$$\partial_x v_y - \partial_y v_x = \frac{1}{2}\mathbf{n}\cdot(\partial_x\mathbf{n}\times\partial_y\mathbf{n}), \quad (64)$$

$$D_t\mathbf{n} = \frac{1}{2}(\mathbf{n}\times\nabla^2\mathbf{n}). \quad (65)$$

With small modifications, these equations can also be shown to describe the dynamics of condensates confined to the ferromagnetic phase of arbitrary spin in the incompressible regime. Our aim is to find stationary solutions of these equations having a skyrmion texture given by^{9,41}

$$\mathbf{n} = [\sin(\beta)\cos(\varphi), \sin(\beta)\sin(\varphi), \cos(\beta)], \quad (66)$$

where φ is the azimuthal angle and β is a function of r which is subject to the boundary conditions $\beta(r=0)=0$ and $\beta(r=R)=\pi$, where R is a distance far from the skyrmion center. Such a spin configuration is shown in Fig. 2.

Given the form \mathbf{n} in Eq. (66), Eqs. (63) and (64) can be solved to obtain the velocity profile. One finds

$$\mathbf{v} = \frac{\sin^2(\beta/2)}{r}\hat{\varphi}. \quad (67)$$

Note that the boundary condition $\beta(0)=0$ suppresses the velocity at the origin which diverges as $1/r$ for the standard $U(1)$ vortex. With the assumption of a static configuration, Eq. (65) reduces to

$$\mathbf{v} \cdot \nabla \mathbf{n} = \frac{1}{2}(\mathbf{n} \times \nabla^2 \mathbf{n}). \quad (68)$$

With the expression for \mathbf{v} in Eq. (67), Eq. (68) leads to the following second-order differential equation for β

$$r \left(r \frac{d^2 \beta}{dr^2} + \frac{d\beta}{dr} \right) = \sin(\beta). \quad (69)$$

With the boundary conditions, the solution of this differential equation is

$$\beta(r) = 4 \tan^{-1}(r/R). \quad (70)$$

This expression, along with the velocity in Eq. (67) and the spin direction in Eq. (66) constitute an analytic stationary solution to the equations of motion for the skyrmion configuration.

V. HYDRODYNAMICS OF SPIN-ONE CONDENSATES

A. Geometrical representation of spin-one hydrodynamic quantities

Now we move on to considering the more complicated case of the spin-one condensate. The spin-one spinor can be broken down into its two spin-half components and be written as $|\chi\rangle = |\Omega\rangle / \sqrt{\langle \Omega | \Omega \rangle}$ where $|\Omega\rangle = |\Omega_1 \Omega_2\rangle$, where we again make use of the large-spin notation defined in Eq. (19). The normalization factor for this case is found to be

$$\langle \Omega | \Omega \rangle = \langle \Omega_1 | \Omega_1 \rangle \langle \Omega_2 | \Omega_2 \rangle + \langle \Omega_1 | \Omega_2 \rangle \langle \Omega_2 | \Omega_1 \rangle = \frac{3}{2} + \frac{1}{2} \mathbf{n}_1 \cdot \mathbf{n}_2. \quad (71)$$

It is also instructive to calculate the spin operator's expectation value. To start we can expand into products of spin-half expectation values

$$\begin{aligned} \langle \Omega | \mathbf{F} | \Omega \rangle &= \langle \Omega_1 | \mathbf{F} | \Omega_1 \rangle + \langle \Omega_2 | \mathbf{F} | \Omega_2 \rangle + \langle \Omega_1 | \Omega_2 \rangle \langle \Omega_2 | \mathbf{F} | \Omega_1 \rangle \\ &+ \langle \Omega_2 | \Omega_1 \rangle \langle \Omega_1 | \mathbf{F} | \Omega_2 \rangle. \end{aligned} \quad (72)$$

Then using the identity in Eq. (36), and the fact that in the factored expression, \mathbf{F} is only acting on spin-half states, we obtain $\langle \Omega | \mathbf{F} | \Omega \rangle = \mathbf{n}_1 + \mathbf{n}_2$. Dividing this by the normalization, we get the spin-one expectation value of the magnetization

$$\mathbf{m} = \langle \chi | \mathbf{F} | \chi \rangle = 2 \frac{\mathbf{n}_1 + \mathbf{n}_2}{3 + \mathbf{n}_1 \cdot \mathbf{n}_2}. \quad (73)$$

By similar techniques, the vector potential for the spin-one case, with some work, can be written as

$$\begin{aligned} a_\alpha &= i \langle \chi | \partial \chi \rangle = \frac{1}{2} \mathbf{e}_{1y} \cdot \partial_\alpha \mathbf{e}_{1x} + \frac{1}{2} \mathbf{e}_{2y} \cdot \partial_\alpha \mathbf{e}_{2x} \\ &+ \frac{1}{2} \frac{(\mathbf{n}_2 \times \mathbf{n}_1) \cdot \partial_\alpha \mathbf{n}_1 + (\mathbf{n}_1 \times \mathbf{n}_2) \cdot \partial_\alpha \mathbf{n}_2}{3 + \mathbf{n}_1 \cdot \mathbf{n}_2}. \end{aligned} \quad (74)$$

One sees that the first two terms in this expression are the vector potentials from the individual spin-half components while the final term, which is gauge invariant, describes their coupling. This expression was previously obtained in Refs.

52 and 53, where a geometrical relation for the Berry phase of a spin-one spinor was given. The field tensor corresponding to this vector potential can also be similarly computed. The most simplified form we find is

$$\begin{aligned} f_{\alpha\beta} &= \frac{-2}{(3 + \mathbf{n}_1 \cdot \mathbf{n}_2)^2} [2\mathbf{n}_1 \cdot (\partial_\alpha \mathbf{n}_1 \times \partial_\beta \mathbf{n}_1) + 2\mathbf{n}_2 \\ &\cdot (\partial_\alpha \mathbf{n}_2 \times \partial_\beta \mathbf{n}_2) + (\mathbf{n}_1 + \mathbf{n}_2) \cdot (\partial_\alpha \mathbf{n}_1 \times \partial_\beta \mathbf{n}_2 + \partial_\alpha \mathbf{n}_2 \\ &\times \partial_\beta \mathbf{n}_1)]. \end{aligned} \quad (75)$$

This is a generalization of the Mermin-Ho relation to the spin-one case. While its geometrical interpretation is not as immediate as the spin-half case (which is the Pontryagin density), this expression might be of use in computing topological invariant quantities for spin-one fields. This formula has a simplified form when locally restricted to mean-field ground states. For instance for the ferromagnetic state ($\mathbf{n} \equiv \mathbf{n}_1 = \mathbf{n}_2$) the above expression reduces to

$$f_{\alpha\beta} = -\mathbf{n} \cdot (\partial_\alpha \mathbf{n} \times \partial_\beta \mathbf{n}). \quad (76)$$

It is also useful to note that for the nematic state ($\mathbf{n} \equiv \mathbf{n}_1 = -\mathbf{n}_2$) the field tensor identically vanishes, $f_{\alpha\beta} = 0$.

Finally, the gauge-invariant quantity Y can be worked out to be

$$\begin{aligned} Y &= \frac{2}{(3 + \mathbf{n}_1 \cdot \mathbf{n}_2)^2} (\partial_\alpha \mathbf{n}_1 \cdot \partial_\alpha \mathbf{n}_1 + \partial_\alpha \mathbf{n}_2 \cdot \partial_\alpha \mathbf{n}_2 + \partial_\alpha \mathbf{n}_1 \cdot \partial_\alpha \mathbf{n}_2 \\ &+ \mathbf{n}_1 \cdot \mathbf{n}_2 \partial_\alpha \mathbf{n}_1 \cdot \partial_\alpha \mathbf{n}_2 - \mathbf{n}_1 \cdot \partial_\alpha \mathbf{n}_2 \mathbf{n}_2 \cdot \partial_\alpha \mathbf{n}_1). \end{aligned} \quad (77)$$

This is an explicit representation of the CP^2 model which can be viewed as a generalization of the nonlinear sigma model. Here, too, it is instructive to consider what this expression reduces to when locally restricting to mean-field ground states. For the ferromagnetic state, one finds

$$Y = \frac{1}{2} \partial_\alpha \mathbf{n} \cdot \partial_\alpha \mathbf{n}. \quad (78)$$

On the other hand, for the nematic state Y reduces to

$$Y = \frac{1}{4} \partial_\alpha \mathbf{n} \cdot \partial_\alpha \mathbf{n}. \quad (79)$$

B. Spin-one condensate equations of motion

We now proceed to do a similar analysis for the spin-one problem. For this we note that the spin-one GP energy functional has the form

$$V_{\text{int}} = \frac{1}{2} g \rho^2 + \frac{1}{2} c_2 \rho^2 \mathbf{m}^2, \quad (80)$$

where \mathbf{m} is the expectation value of the spin-one operator. The first two hydrodynamic equations—the mass continuity equation and the modified Euler equation—are obtained, as before, by contracting the Gross-Pitaevskii equation with $\langle \chi |$. The analysis proceeds along similar lines as the spin-half case. However, for this case we need the generalization of the Mermin-Ho relation for spin one given in Eq. (75) to

give the field tensor and thus the effective electric and magnetic fields, in addition to the spin-one expressions for \mathbf{Y} Eq. (77) and the magnetization \mathbf{m} , in Eq. (73). With these quantities, the first two equations of motion are

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) \quad (81)$$

and

$$D_t \mathbf{v} = \boldsymbol{\epsilon} + (\mathbf{v} \times \mathbf{b}) - \nabla \left(g\rho + c_2 \rho m^2 + \frac{1}{2} \mathbf{Y} - \frac{\nabla^2 \sqrt{\rho}}{2\sqrt{\rho}} \right). \quad (82)$$

Next, let us discuss the spin dynamical equations. These are obtained by contracting the GPE with $\langle \Omega_1^t \Omega_1^t |$ and $\langle \Omega_2^t \Omega_2^t |$. As before, this causes several terms to vanish since these spinors are orthogonal to $|\chi\rangle$. Contracting with $\langle \Omega_1^t \Omega_1^t |$ gives the following equation which gives the time derivative of the first node

$$i\mathbf{e}_{1+} \cdot D_t \mathbf{n}_1 = -\frac{1}{2} \Gamma_\alpha^{12} \mathbf{e}_{1+} \cdot \partial_\alpha \mathbf{n}_1 - \frac{1}{2} \mathbf{e}_{1+} \cdot \nabla^2 \mathbf{n}_1 + c_2 \rho \mathbf{e}_{1+} \cdot \mathbf{m}. \quad (83)$$

A similar equation for the time derivative of \mathbf{n}_2 is obtained by contracting with $\langle \Omega_2^t \Omega_2^t |$. In the above, we have collected the following terms into the Γ_α^{ij} parameter

$$\Gamma_\alpha^{ij} = 2 \frac{\partial_\alpha \sqrt{\rho}}{\sqrt{\rho}} - \frac{\partial_\alpha (\mathbf{n}_i \cdot \mathbf{n}_j)}{3 + \mathbf{n}_i \cdot \mathbf{n}_j} - \frac{\mathbf{n}_i \cdot \partial_\alpha \mathbf{n}_j - i \mathbf{n}_i \cdot (\mathbf{n}_j \times \partial_\alpha \mathbf{n}_j)}{1 - \mathbf{n}_i \cdot \mathbf{n}_j} \quad (84)$$

$$+ i \frac{(\mathbf{n}_j \times \mathbf{n}_i) \cdot \partial_\alpha \mathbf{n}_i + (\mathbf{n}_i \times \mathbf{n}_j) \cdot \partial_\alpha \mathbf{n}_j}{3 + \mathbf{n}_i \cdot \mathbf{n}_j}. \quad (85)$$

Finally, separating the real and imaginary parts as $\Gamma_\alpha^{ij} = (\Gamma_\alpha^{ij})' + i(\Gamma_\alpha^{ij})''$, we obtain the Landau-Lifshitz equations

$$\left[D_t + \frac{1}{2} (\Gamma_\alpha^{12})'' \partial_\alpha \right] \mathbf{n}_1 = \frac{1}{2} \mathbf{n}_1 [(\Gamma_\alpha^{12})' \partial_\alpha \mathbf{n}_1 + \nabla^2 \mathbf{n}_1] - c_2 \rho \mathbf{n}_1 \mathbf{m}, \quad (86)$$

$$\left[D_t + \frac{1}{2} (\Gamma_\alpha^{21})'' \partial_\alpha \right] \mathbf{n}_2 = \frac{1}{2} \mathbf{n}_2 [(\Gamma_\alpha^{21})' \partial_\alpha \mathbf{n}_2 + \nabla^2 \mathbf{n}_2] - c_2 \rho \mathbf{n}_2 \mathbf{m}. \quad (87)$$

This provides a complete set of equations describing the dynamics of the spin-one condensate.

VI. HYDRODYNAMICS FOR GENERAL SPIN- F CONDENSATES

Now that we have considered the hydrodynamic equations for spin-half and spin-one condensates in detail, in the following we will consider the general case. The first two equations of motion, the mass continuity equation, and the Euler equation are found, as before, to be

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) \quad (88)$$

and

$$D_t \mathbf{v} = \boldsymbol{\epsilon} + (\mathbf{v} \times \mathbf{b}) - \nabla \left(\frac{2V_{\text{int}}}{\rho} + \frac{1}{2} \mathbf{Y} - \frac{\nabla^2 \sqrt{\rho}}{2\sqrt{\rho}} \right). \quad (89)$$

The effective electric and magnetic fields again follow from the field tensor $f_{\alpha\beta}$ constructed from $a_\alpha = i\langle \chi | \partial_\alpha \chi \rangle$. For a general spin, however, such quantities are cumbersome to express directly in terms of the spin nodes, and we will refrain from doing so.

To obtain the Landau-Lifshitz equations, we contract the GPE with $\langle (\Omega_i^t)^{2F} |$. Doing this gives

$$\begin{aligned} i\langle \Omega_i^t | \partial_t \Omega_i \rangle &= -\partial_\alpha \log \left(\frac{\psi}{\sqrt{\langle \Omega | \Omega \rangle}} \right) \langle \Omega_i^t | \partial_\alpha \Omega_i \rangle - \frac{1}{2} \langle \Omega_i^t | \nabla^2 \Omega_i \rangle \\ &\quad - \langle \Omega_i^t | \partial_\alpha \Omega_i \rangle \sum_{j \neq i} \frac{\langle \Omega_j^t | \partial_\alpha \Omega_j \rangle}{\langle \Omega_j^t | \Omega_j \rangle} \\ &\quad + \frac{\rho}{\lambda_i^* \langle \Omega | \Omega \rangle} {}_1 \langle (\Omega_i^t)^{2F} | {}_2 \langle \Omega | \mathcal{V}_{\text{int}} | \Omega \rangle {}_2 | \Omega \rangle {}_1. \end{aligned} \quad (90)$$

In this expression, we have used the notation for interaction energy introduced in Eq. (10). In addition we have introduced the quantities λ_i

$$\lambda_i = (2F)! \prod_{j \neq i} \langle \Omega_j | \Omega_j^t \rangle. \quad (91)$$

While the first term in Eq. (90)

$$i\langle \Omega_i^t | \partial_t \Omega_i \rangle = \frac{i}{2} \mathbf{e}_{i+} \cdot \partial_t \mathbf{n}_i$$

is the inertial term for the spin node \mathbf{n}_i , the right-hand side, and the last term of Eq. (90) in particular, should serve the role of torques, projected onto \mathbf{e}_{i+} . As we will show in the next section, the matrix element of \mathcal{V}_{int} is indeed related to a derivative with respect to the spin-node coordinates of a potential-energy function. Specifically

$$\begin{aligned} \frac{\rho}{\lambda_i^* \langle \Omega | \Omega \rangle} {}_1 \langle (\Omega_i^t)^{2F} | {}_2 \langle \Omega | \mathcal{V}_{\text{int}} | \Omega \rangle {}_2 | \Omega \rangle {}_1 &= A_{ij}^{-1} \rho \langle \Omega | \Omega \rangle \\ &\quad \times [\mathbf{e}_{j+} \cdot \nabla_{\mathbf{n}_j} V(\{\mathbf{n}_j\})], \end{aligned} \quad (92)$$

where $V(\{\mathbf{n}_j\}) = \langle \mathcal{V}_{\text{int}} \rangle$ is the expectation value of the energy of a spin configuration with spin nodes $\{\mathbf{n}_j\}$, and A_{ij}^{-1} is a matrix which projects the torques due to spin node j and the motion of spin node i . The matrix A and its inverse are defined below in Eq. (121).

Instead of writing Eq. (90) in terms of vectors as in the previous sections, we will stop at this point. This equation provides a natural starting point in the analysis of the linearized equations of motion which will be developed in the following section.

VII. LINEARIZED EQUATIONS OF MOTION FOR ARBITRARY SPIN- F CONDENSATES

As suggested from the equations of motion of the spin-one and higher condensates given in the previous sections, the geometric representation of the equations of motion yield

rather complicated results. Nevertheless, this formalism regains its appeal when linearized about particular mean-field ground states. Then the hidden point symmetries of the ground state become apparent and can be used to describe the linearized dynamics of a condensate. Below we derive the small oscillation description of general spinor condensates.

A. Linearized equations of motion from the GPE

Parting ways from the attempt at a general description of spinor-condensate dynamics, we now turn to the vicinity of a uniform mean-field ground state. For the ensuing discussion, we will denote quantities to be evaluated in the mean-field ground state with overhead bars. For instance, the density can be written by expanding about the mean-field state as

$$\rho = \bar{\rho} + \delta\rho. \quad (93)$$

We will first concentrate on the equations describing the density excitations. Linearizing the equations of motion derived in Sec. VI leads to the following two equations describing the density fluctuations:

$$\partial_t \rho = -\bar{\rho} \nabla \cdot \mathbf{v} \quad (94)$$

and

$$\partial_t \mathbf{v} = -\nabla \left(-\frac{2\bar{V}_{\text{int}}}{\bar{\rho}^2} \rho + \frac{\nabla^2 \rho}{2\bar{\rho}} \right). \quad (95)$$

Note that terms describing the spin degrees of freedom (e.g., the effective electric and magnetic fields) have completely dropped out of these equations from linearization. Computing the excitations from these equations is straightforward and gives the familiar Bogoliubov mode describing density fluctuations.

Let us now focus our attention on linearizing the Landau-Lifshitz equations for general spin written in Eq. (90). Since the process of linearization separates the equations for spin and density fluctuations, to simplify the notation in what follows, we will scale the density of the uniform state to one, $\rho_0 \rightarrow 1$. When linearized, the Landau-Lifshitz equations for general spin become

$$\begin{aligned} i\langle \bar{\Omega}_i^f | \partial_t \Omega_i \rangle &= -\frac{1}{2} \langle \bar{\Omega}_i^f | \nabla^2 \Omega_i \rangle \\ &+ \frac{1}{\lambda_i^* \langle \bar{\Omega} | \Omega \rangle} \langle (\bar{\Omega}_i^f)^{2F} | {}_2 \langle \Omega | \mathcal{V}_{\text{int}} | \Omega \rangle_2 | \Omega \rangle_1. \end{aligned} \quad (96)$$

In the above, as before, we have used overhead bars to denote quantities evaluated at their mean-field configuration.

To understand the dynamics of Eq. (96) it is useful to introduce variables to describe small deviations of the spin nodes from their mean-field values. To this end, by using the identities established in Sec. III E, we introduce the set of $2F$ complex variables $\{z_i\}$

$$z_i \equiv \langle \bar{\Omega}_i^f | \Omega_i \rangle = \langle \bar{\Omega}_i^f | \delta \Omega_i \rangle = \frac{1}{2} \bar{\mathbf{e}}_{i+} \cdot \mathbf{n}_i = \frac{1}{2} \bar{\mathbf{e}}_{i+} \cdot \delta \mathbf{n}_i, \quad (97)$$

where $\mathbf{n}_i = \bar{\mathbf{n}}_i + \delta \mathbf{n}_i$. Note that in the mean-field states we have $\bar{z}_i = 0$ for each spin node since the vectors $\bar{\mathbf{e}}_{i+}$ and $\bar{\mathbf{n}}_i$ are orthogonal. This set of variables can be seen to be the local stereographic projection of \mathbf{n}_i onto the complex plane for small displacements and will be very useful in the following analysis. Moreover, in our gauge convention, z_i is given in terms of displacements along the zenith and azimuthal directions from the spherical coordinate system

$$z_i = \delta \mathbf{n} \cdot \hat{\theta} + i \delta \mathbf{n} \cdot \hat{\phi}. \quad (98)$$

Using these variables, the linearized Landau-Lifshitz equations become

$$i \partial_t z_i = -\frac{1}{2} \nabla^2 z_i + \frac{{}_1 \langle (\bar{\Omega}_i^f)^{2F} | {}_2 \langle \Omega | \mathcal{V}_{\text{int}} | \Omega \rangle_2 | \Omega \rangle_1}{\lambda_i^* \langle \bar{\Omega} | \Omega \rangle}. \quad (99)$$

The kinetic pieces in the GP equations are most naturally described in terms of the original spinor wave function, ψ_a , and are not simplified by the symmetry of the mean-field ground states. Nevertheless, Eq. (99) demonstrates that near mean-field ground states the kinetic terms still acquire a simple form. Interestingly, the kinetic parts in the spin equations of motion, Eq. (99), do not disclose the fact that the variables $\{z_i\}$ describe spin-half components of a spin- F state. This fact is reflected only in the spin-interaction term. In the following section, we will see that this spin interaction can be expressed in terms of a derivative with respect to the z^* variables. In particular, the equations of motion will be shown to be

$$i \partial_t z_i = -\frac{1}{2} \nabla^2 z_i + \langle \bar{\Omega} | \bar{\Omega} \rangle \sum_j \bar{A}_{ij}^{-1} \frac{\partial}{\partial z_j^*} V_{\text{int}}, \quad (100)$$

where

$$\bar{A}_{ij}^{-1} \equiv \frac{\langle (\bar{\Omega}_i^f)^{2F} | (\bar{\Omega}_j^f)^{2F} \rangle}{\lambda_i^* \lambda_j}. \quad (101)$$

Thus, the spin interaction derives from a sum over ‘‘torques’’

$$\tau_j = \frac{\partial}{\partial z_j^*} V_{\text{int}}. \quad (102)$$

B. Perturbative expansion of the spin interaction

An essential element in the behavior of spinor condensates is the spin-interaction term V_{int} . It is the minimization of this term that yields the mean-field ground states and its curvature that determines the normal excitations. These curvatures can be easily and directly extracted in terms of specific matrix elements, as we show below.

To expand the spin-interaction energy about a mean-field ground state (denoted with an overhead bar) we first need to understand how to perturb a spinor about a fixed value. The following spin-half identity proves to be quite helpful:

$$|\delta\Omega\rangle = |\bar{\Omega}\rangle\langle\bar{\Omega}|\delta\Omega\rangle + |\bar{\Omega}_t\rangle_z, \quad (103)$$

where we used the resolution of the identity in terms of $|\bar{\Omega}\rangle$ and its time-reversed partner, and the definition of $z = \langle\bar{\Omega}_t|\delta\Omega\rangle$ as in Eq. (97). Now, if we apply the variation to a general spin- F spinor $|\Omega\rangle = |\bar{\Omega}\rangle + \delta|\Omega\rangle$, we obtain to linear order

$$\delta|\Omega\rangle = |\bar{\Omega}\rangle \sum_{i=1}^{2F} \langle\bar{\Omega}_i|\delta\Omega_i\rangle + \sum_{i=1}^{2F} |T_i\bar{\Omega}\rangle_{z_i}, \quad (104)$$

where $|T_i\bar{\Omega}\rangle$ is $|\bar{\Omega}\rangle$ with its i th entry time reversed (see Appendix A). Since $\langle\bar{\Omega}_i|\delta\Omega_i\rangle$ is imaginary, the first term, which does not directly depend on z , must drop off when considering the variations of real quantities. For instance, the first-order variation of the normalization is

$$\delta\langle\Omega|\Omega\rangle = \sum_{i=1}^{2F} (\langle\bar{\Omega}|T_i\bar{\Omega}\rangle_{z_i} + \langle T_i\bar{\Omega}|\bar{\Omega}\rangle_{z_i}^*). \quad (105)$$

Using Eqs. (104) and (105) one finds

$$\frac{\partial}{\partial z_i} \langle\Omega|\Omega\rangle = \frac{\bar{\mathcal{P}}|T_i\bar{\Omega}\rangle}{\langle\Omega|\Omega\rangle}, \quad (106)$$

where

$$\bar{\mathcal{P}} = 1 - \frac{|\bar{\Omega}\rangle\langle\bar{\Omega}|}{\langle\bar{\Omega}|\bar{\Omega}\rangle}. \quad (107)$$

Such an expression is useful in evaluating derivatives of the spin-interaction energy as in Eq. (100). In general, derivatives with respect to z_i^* will act on bras while derivatives with respect to z_i will act on kets.

We will now establish the equivalence between Eqs. (99) and (100). One can use Eq. (106) to evaluate the derivative of the interaction energy

$$\frac{\partial}{\partial z_j^*} V_{\text{int}} = \frac{{}_1\langle T_j\bar{\Omega}|\bar{\mathcal{P}}_{12}\langle\Omega|\mathcal{V}_{\text{int}}|\Omega\rangle_2|\Omega\rangle_1}{\langle\Omega|\Omega\rangle^2} \quad (108)$$

which is correct to linear order. The subscripts of the bra's and ket's denote how the inner product is to be evaluated: ket 1 (2) is contracted with bra 1 (2), and signify the state of one of two interacting particles; similarly, the projection $\bar{\mathcal{P}}_1$ operates only on the degrees of freedom pertaining to particle "1." Then using the expression for A^{-1} and the relation (derived in Appendix B)

$$\mathcal{P} = \sum_i \frac{|(\Omega_i^t)^{2F}\rangle\langle T_i\Omega|}{\lambda_i} \mathcal{P} \quad (109)$$

one immediately finds for the last term in Eq. (100)

$$\langle\bar{\Omega}|\bar{\Omega}\rangle \sum_j \bar{A}_{ij}^{-1} \frac{\partial}{\partial z_j^*} V_{\text{int}} = \frac{{}_1\langle (\bar{\Omega}_i^t)^{2F}|_2\langle\Omega|\mathcal{V}_{\text{int}}|\Omega\rangle_2|\Omega\rangle_1}{\lambda_i^* \langle\bar{\Omega}|\bar{\Omega}\rangle} \quad (110)$$

which is the last term in Eq. (99).

I. Second-order expansion of the interaction energy

Since we are interested in small oscillations about equilibrium, we would like to express the interaction energy expanded about the mean-field state to quadratic order in the z variables. This can be formally written as

$$V_{\text{int}} = \bar{V}_{\text{int}} + \frac{1}{2} \sum_{ij} \frac{\partial^2 V_{\text{int}}}{\partial z_i \partial z_j} z_i z_j + \sum_{ij} \frac{\partial^2 V_{\text{int}}}{\partial z_i^* \partial z_j} z_i^* z_j, \quad (111)$$

$$+ \frac{1}{2} \sum_{ij} \frac{\partial^2 V_{\text{int}}}{\partial z_i^* \partial z_j^*} z_i^* z_j^*, \quad (112)$$

where the terms involving derivatives of V_{int} are to be evaluated at the mean-field ground state. We can now use Eq. (106) to evaluate these derivatives of the interaction energy. Note that terms where two derivatives act on the same bra or ket will vanish since

$$\bar{\mathcal{P}}_{12} \langle\bar{\Omega}|\mathcal{V}_{\text{int}}|\bar{\Omega}\rangle_2 |\bar{\Omega}\rangle_1 = 0 \quad (113)$$

which happens since $\tau_i=0$ at the minimum of the spin interaction, so that ${}_2\langle\bar{\Omega}|\mathcal{V}_{\text{int}}|\bar{\Omega}\rangle_2 |\bar{\Omega}\rangle_1 \propto |\bar{\Omega}\rangle_1$. We then readily obtain the following quadratic form for the spin-interaction energy (dropping the \bar{V}_{int} term):

$$V_{\text{int}} = \sum_{ij} \left(\frac{{}_1\langle\bar{\Omega}|_2\langle\bar{\Omega}|\mathcal{V}_{\text{int}}\bar{\mathcal{P}}_2|T_i\bar{\Omega}\rangle_2 \bar{\mathcal{P}}_1|T_j\bar{\Omega}\rangle_1}{2\langle\bar{\Omega}|\bar{\Omega}\rangle^2} z_i z_j \right. \\ \left. + \frac{{}_1\langle T_i\bar{\Omega}|\bar{\mathcal{P}}_{12}\langle\bar{\Omega}|\mathcal{V}_{\text{int}}|\bar{\Omega}\rangle_2 \bar{\mathcal{P}}_1|T_j\bar{\Omega}\rangle_1}{\langle\bar{\Omega}|\bar{\Omega}\rangle^2} z_i^* z_j \right. \\ \left. + \frac{{}_1\langle T_i\bar{\Omega}|\bar{\mathcal{P}}_{12}\langle T_j\bar{\Omega}|\bar{\mathcal{P}}_2\mathcal{V}_{\text{int}}|\bar{\Omega}\rangle_2 |\bar{\Omega}\rangle_1}{2\langle\bar{\Omega}|\bar{\Omega}\rangle^2} z_i^* z_j^* \right). \quad (114)$$

Here $\bar{\mathcal{P}}_{1,2}$ is the projection operator which only acts on states denoted with subscripts 1 or 2, respectively. While the form above is written symmetrically, following Eq. (113), only one projector is needed in Eq. (114), so \mathcal{P}_2 can be omitted.

While these results for the spin interaction seem involved, they are directly expressed in terms of easily constructed matrix elements evaluated at the mean-field ground state. Furthermore, these matrix elements obey the point symmetry of the ground state at hand and thus have stringent constraints. Eq. (114) therefore provides us with direct expressions for the matrix elements appearing in the linear spin-wave expansion of the spinor condensate.

C. The Lagrangian of spinor condensates near equilibrium

The equations of motion can be arrived at by expanding the spinor-condensate Lagrangian to quadratic order in the z variables and computing the corresponding Euler-Lagrange equations. As we saw before, to this order, the density excitations decouple from the spin excitations. Thus, to simplify the analysis, we will fix the density and scale it to one and work in the incompressible regime. The Lagrangian for a spin- F condensate in the incompressible regime is

$$\mathcal{L} = a_t - \frac{1}{2}(\nabla\theta - \mathbf{a})^2 - \frac{1}{2}Y - V_{\text{int}}, \quad (115)$$

where V_{int} is the spin-interaction potential. In expanding this Lagrangian to second order, we first consider the spin Berry's phase contribution

$$a_t = i\langle\chi|\partial_t\chi\rangle = \frac{i}{2} \frac{\langle\mathbf{\Omega}|\partial_t\mathbf{\Omega}\rangle - \langle\partial_t\mathbf{\Omega}|\mathbf{\Omega}\rangle}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle}. \quad (116)$$

Note that the kets and bras involving time derivatives are necessarily first order in variation from the mean-field state. Thus we consider the following quantity expanded to first order about the ground state

$$\delta \frac{|\mathbf{\Omega}\rangle}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle} = \frac{|\delta\mathbf{\Omega}\rangle}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle} - \frac{|\overline{\mathbf{\Omega}}\rangle}{\langle\overline{\mathbf{\Omega}}|\mathbf{\Omega}\rangle} \delta\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle, \quad (117)$$

$$= \frac{\overline{\mathcal{P}}|\delta\mathbf{\Omega}\rangle}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle} - |\overline{\mathbf{\Omega}}\rangle \frac{\langle\delta\mathbf{\Omega}|\overline{\mathbf{\Omega}}\rangle}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle^2}. \quad (118)$$

Inserting this into the expression for the spin Berry's phase (116), and dropping terms that can be written as total time derivatives (which do not contribute to the dynamics) one finds

$$a_t = i \frac{\langle\mathbf{\Omega}|\overline{\mathcal{P}}|\partial_t\mathbf{\Omega}\rangle}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle}. \quad (119)$$

We can then insert into Eq. (119) the expressions for the expansion of $|\mathbf{\Omega}\rangle$ to linear order in the z variables given in Eq. (104) to directly obtain

$$a_t = \frac{i}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle} \sum_{ij} z_i^* \bar{A}_{ij} \partial_t z_j, \quad (120)$$

where

$$A_{ij} \equiv \langle T_i \mathbf{\Omega} | \overline{\mathcal{P}} | T_j \mathbf{\Omega} \rangle \quad (121)$$

which is the sought-after relation. The proof that A defined here is in fact the inverse of the expression given in Eq. (101) is given in Appendix B. The hermitian matrix \bar{A} gives the canonical commutation relations between the z variables. To directly compute the matrix elements of A is cumbersome because each involves a Wick expansion of $(2F)!$ terms. On the other hand the expression for A^{-1} given in Eq. (101) is readily computed since it involves evaluating overlaps between spin-coherent states. Thus, in practice, to construct the matrix A it is easiest to first construct A^{-1} and then compute its inverse.

Proceeding along very similar lines as above, one can expand Y to second order in the z 's. One finds

$$Y = \langle\partial_\alpha\chi|\overline{\mathcal{P}}|\partial_\alpha\chi\rangle \approx \frac{1}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle} \sum_{ij} \partial_\alpha z_i^* \bar{A}_{ij} \partial_\alpha z_j. \quad (122)$$

Finally, we note that the term involving the superfluid velocity $\mathbf{v} = \nabla\theta - \mathbf{a}$ in the Lagrangian will not contribute to the linearized equations of motion. We are now in a position to vary the Lagrangian Eq. (115) as a function of the z 's to find the linearized equations of motion. These read

$$i\bar{A}_{ij}\partial_t z_j = -\frac{1}{2}\bar{A}_{ij}\nabla^2 z_j + \langle\overline{\mathbf{\Omega}}|\mathbf{\Omega}\rangle \frac{\partial V}{\partial z_i^*} \quad (123)$$

(repeated indices are summed over). It is straightforward to see that this is the same as Eq. (100) which was obtained directly from linearizing the GPE contracted with time-reversed coherent states.

Since A is a hermitian matrix, it is diagonalized by a unitary transformation

$$A = U\Lambda U^\dagger, \quad (124)$$

where Λ is the diagonal matrix consisting of the eigenvalues of A . It is therefore convenient to define a new set of w coordinates as

$$w = \bar{U}^\dagger z. \quad (125)$$

Note that in terms of these coordinates, the Berry's phase assumes a simple diagonal form

$$a_t = \frac{1}{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle} \sum_i \bar{\Lambda}_i w_i^* \partial_t w_i. \quad (126)$$

Furthermore, the equations of motion have the simple form in these coordinates

$$i\partial_t w_i = -\frac{1}{2}\nabla^2 w_i + \frac{\langle\overline{\mathbf{\Omega}}|\mathbf{\Omega}\rangle}{\bar{\Lambda}_i} \frac{\partial V}{\partial w_i^*}. \quad (127)$$

This has the form of a time-dependent Schrodinger equation for the w_i parameters.

VIII. NORMAL EIGENMODES, SYMMETRY, AND GROUP THEORY

The most appealing application of the linearized equations of motion developed in the previous section is to obtain the normal excitation modes and energies of spinor condensates having a hidden ground-state symmetry. As we show, it is nearly sufficient to diagonalize the matrix A [defined in Eq. (101)] in order to obtain the eigenmodes of the spinor condensate. This can be done solely by using the symmetry of the mean-field state.

Below we first demonstrate the use of the linearized equations of motion on the cyclic state without fully utilizing the symmetry in Sec. VIII A and obtain all eigenmodes and eigenfrequencies using the variables defined in Sec. VII B 1. Next, in Sec. VIII B, we demonstrate how from the point group of the hidden symmetry of the mean-field ground states, we can compute the normal modes alone (but not energies), using the example of the spin-three state where the spin nodes are arranged at the vertices of a hexagon. Finally, in Sec. VIII C, we show how to directly construct the vibrational and rotational eigenmodes from spherical harmonics, by connecting the problem at hand to that of degeneracy lifting of electronic atomic orbitals. This method circumvents the arduous group-theory foot work, by using the well-known properties of atomic orbitals under crystal fields that break rotational invariance.

The general motivation of the discussion below is that group-theory analysis can be applied to obtain the normal

modes in spinor condensates much like the analysis of the vibrational frequencies of polyatomic molecules.^{35,36} The “atoms” (or spin nodes) in our case, however, are confined to the surface of the unit sphere, and the displacement of each spin node is a two-dimensional vector (parameterized by the real and imaginary parts of the z variables). This constraint slightly complicates the analysis in comparison with the treatment of the vibrations of polyatomic molecules. The first step in a symmetry analysis is to construct the transformation rules of the two-dimensional displacement vectors under point-group symmetries. These transformation rules are a reducible representation of the symmetry group and can then be broken down into its irreducible representations (irreps). The modes that transform according to the irreps are the eigenmodes of the system.

Before we begin the analysis, a note on mode multiplicity is in order. Naively, one might expect that the procedure in the previous paragraph will give $(2F) \times 2$ normal modes due to the two basis vectors per spin node. This situation would arise if the transformations we construct transform the $2F \times 2$ *real* coordinates, and are therefore $4F$ large real reducible representations of the symmetry group, resulting in $4F$ modes. While this is the case for real atoms, where the displacement vectors are also associated with conjugate momenta, the spin-nodes displacements do not have independent conjugate momenta. From Eqs. (115) and (116) we see that the complex displacement z_i is actually canonically conjugate to $\pi_i = \frac{\partial \mathcal{L}}{\partial z_i} \propto i \sum_j A_{ij} z_j^*$; the two-dimensional displacements are both the coordinate and conjugate momenta and hence there are only $2F$ eigenmodes in a spinor condensate. Qualitatively, this is a situation reminiscent of a massless particle in a magnetic field, where the x and y coordinates are canonically conjugate coordinate and momentum. Indeed, constructing real $4F$ dimensional representations of the symmetry would result in two duplicates of the spinor-condensate’s eigenmodes. This duplicity will become evident when the eigenmodes are written in terms of the complex z_i ’s: half the normal modes will differ from the other half through a complex multiplicative coefficient.

A. Spin-two cyclic state

As our first example, we consider the cyclic state which is a possible mean-field ground state having the symmetry of a tetrahedron for the spin-two problem. We will expand the interaction energy to quadratic order about this mean-field ground state to compute the energies of the normal excitations. The spin-two interaction energy can be written in the simple form^{28,29}

$$V_{\text{int}} = \frac{1}{2} \alpha m^2 + \frac{1}{2} \beta |\langle \chi_l | \chi \rangle|^2, \quad (128)$$

where α and β are functions of the scattering lengths, and

$$\mathbf{m} = \langle \chi | \mathbf{F} | \chi \rangle. \quad (129)$$

For the mean-field cyclic state, this spin-interaction energy conveniently vanishes $\bar{V}_{\text{int}} = 0$. In the following we will expand this energy to quadratic order.

We first construct the symmetry matrix A for the cyclic state. We take the orientation where the spin nodes are at (in Cartesian coordinates)

$$\bar{\mathbf{n}}_1 = \frac{1}{\sqrt{3}}(1, 1, 1), \quad \bar{\mathbf{n}}_2 = \frac{1}{\sqrt{3}}(-1, -1, 1), \quad (130)$$

$$\bar{\mathbf{n}}_3 = \frac{1}{\sqrt{3}}(1, -1, -1), \quad \bar{\mathbf{n}}_4 = \frac{1}{\sqrt{3}}(-1, 1, -1). \quad (131)$$

With the spin-half spinors corresponding to these spin nodes the matrix \bar{A}^{-1} can be directly constructed using the expression involving overlaps of time-reversed coherent states in Eq. (101). Using our gauge convention, this is found to be

$$\bar{A}^{-1} = \frac{1}{64} \begin{pmatrix} 9 & 1 & -1 & -1 \\ 1 & 9 & -1 & -1 \\ -1 & -1 & 9 & 1 \\ -1 & -1 & 1 & 9 \end{pmatrix}. \quad (132)$$

This then can be inverted to obtain

$$\bar{A} = \frac{2}{3} \begin{pmatrix} 11 & -1 & 1 & 1 \\ -1 & 11 & 1 & 1 \\ 1 & 1 & 11 & -1 \\ 1 & 1 & -1 & 11 \end{pmatrix}. \quad (133)$$

Recall that directly constructing the \bar{A} matrix is cumbersome since its elements involve Wick expansions having $(2F)!$ terms. The eigenvalues of this matrix are found to be $\text{Eig}(\bar{A}) = (\bar{\Lambda}_1, \bar{\Lambda}_2, \bar{\Lambda}_3, \bar{\Lambda}_4) = (8, 8, 8, \frac{16}{3})$. This matrix can be written in a revealing form as

$$\bar{A} = 8I - \frac{8}{3} \bar{u}_4 \bar{u}_4^\dagger, \quad (134)$$

where $\bar{u}_4 = \frac{1}{2}(1, 1, -1, -1)^T$ is the eigenvector of \bar{A} corresponding to eigenvalue $\bar{\Lambda}_4$ and I is the identity matrix. An eigenmode will necessarily diagonalize the A matrix as well as the entire equations of motion and therefore we already gleaned one eigenmode: \bar{u}_4 , which will turn out to be the optical mode.

The three modes orthogonal to \bar{u}_4 are associated with $SO(3)$ rotations. With this in mind, we construct these three eigenmodes as the vectors arising from infinitesimal rotations of $\bar{\mathbf{n}}_i$ about the Cartesian axes, $\hat{\mathbf{x}}_\alpha$. A rotation by angle $\delta\eta$ about the $\hat{\mathbf{x}}_\alpha$ axis produces the following z_i ’s:

$$z_i(\delta\eta) = \delta\eta(\hat{\mathbf{x}}_\alpha \times \bar{\mathbf{n}}_i) \cdot \bar{\mathbf{e}}_{i+} = i\delta\eta \bar{\mathbf{e}}_{i+} \cdot \hat{\mathbf{x}}_\alpha. \quad (135)$$

Thus the eigenvectors \bar{u}_α are

$$\bar{u}_\alpha = \sqrt{\frac{3}{8}} \{\bar{\mathbf{e}}_{i+} \cdot \hat{\mathbf{x}}_\alpha\}_{i=1}^4. \quad (136)$$

It is now clear how to write the transformation into the eigencoordinates defined generally in Eq. (125)

$$z_i = \sum_{\alpha} w_{\alpha} (\bar{u}_{\alpha})_i. \quad (137)$$

Due to the high symmetry of the tetrahedron, these modes are also degenerate. In general, the set of coordinate vectors $\hat{\mathbf{x}}_{\alpha}$ should be taken to be the principal axes of the mean-field configuration.

Next we use this matrix to expand the interaction energy. We first consider the linear order variation of the spin moment \mathbf{m} . Note that since $\bar{\mathbf{m}}=0$ in the ground state we have $\langle \bar{\mathbf{\Omega}} | \mathbf{F} | \bar{\mathbf{\Omega}} \rangle = \langle \bar{\mathbf{\Omega}} | \mathbf{F} \bar{\mathcal{P}} | \bar{\mathbf{\Omega}} \rangle$. Then by inserting the identity for $\bar{\mathcal{P}}$ given in Eq. (109) one finds

$$\delta \mathbf{m} = \frac{1}{2 \langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle} \sum_{ij} (\bar{\mathbf{e}}_{i-} \bar{A}_{ij} z_j + z_i^* \bar{A}_{ij}^* \bar{\mathbf{e}}_{j+}). \quad (138)$$

We now write the vectors $\bar{\mathbf{e}}_{i\pm}$ in the basis of unit vectors along the three Cartesian coordinates

$$\bar{\mathbf{e}}_{i\pm} = \sum_{\alpha=1}^3 (\bar{\mathbf{e}}_{i\pm} \cdot \hat{\mathbf{x}}_{\alpha}) \hat{\mathbf{x}}_{\alpha} \quad (139)$$

which immediately reduces them to the complex conjugate of the degenerate eigenvectors \bar{u}_{α} (with eigenvalue $\bar{\Lambda}=8$). The fact that all of the eigenvalues are the same is due to the high symmetry of the tetrahedral state. With this basis one finds for the expansion of magnetization the simple expression

$$\begin{aligned} \delta \mathbf{m} &= \frac{4}{\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle} \sum_i \sum_{\alpha=1}^3 \sqrt{\frac{8}{3}} \hat{\mathbf{x}}_{\alpha} ((\bar{u}_{\alpha})_i^* z_i + (\bar{u}_{\alpha})_i z_i^*) \\ &= \sqrt{6} \sum_{\alpha=1}^3 \hat{\mathbf{x}}_{\alpha} (w_{\alpha} + w_{\alpha}^*), \end{aligned} \quad (140)$$

where we have expressed the final result in terms of the w variables [defined in Eq. (137)]. In deriving the above expression, we have explicitly used the values for the eigenvalues of the A matrix and the normalization constant $\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle = \frac{8}{3}$. The three parameters of w_i occurring in Eq. (140) correspond to rotations about the three Cartesian axes as shown in Fig. 3.

Similar analysis can be performed on the second term in the spin interaction for the cyclic state. Without showing the details, it is found that

$$\delta \langle \chi_i | \chi \rangle = 2\sqrt{2} w_4. \quad (141)$$

With these expressions we can now write down the spin-interaction energy expanded to quadratic order which reads

$$\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle V_s = \alpha \sum_{i=1}^3 \bar{\Lambda}_i (w_i + w_i^*)^2 + 2\beta \bar{\Lambda}_4 |w_4|^2. \quad (142)$$

With this expansion of the interaction, Eq. (127) can be directly used to compute the energy of the normal excitations. Four Bogoliubov modes (note we are neglecting the density mode) are readily obtained. One finds three gapless spin waves of dispersion $E_k^s = \sqrt{\varepsilon_k(\varepsilon_k + 4\alpha)}$ in addition to an optical mode having dispersion $E_k^{op} = \varepsilon_k + 2\beta$ (where ε_k is the free particle dispersion).

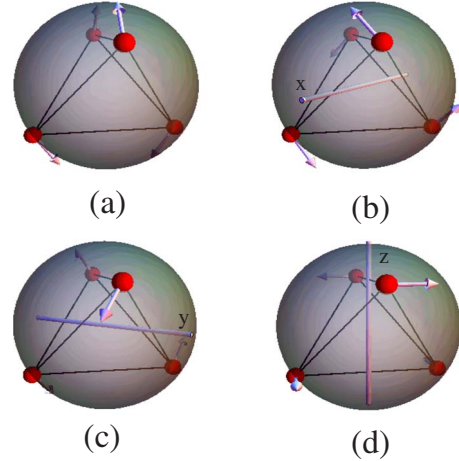


FIG. 3. (Color online) Normal modes of the cyclic state. Mode (a) is the optical mode corresponding to pure displacements in w_4 . Modes (b), (c), and (d) are gapless modes corresponding rotating about the x , y , or z axes, respectively. The axes of rotation for these modes are shown.

Quite generally, the eigenvectors of the matrix \bar{A} yield the displacements of the z variables corresponding to each of the eigenmodes [see, e.g., Eq. (135)]. In case of degeneracy, it is the interaction terms, discussed in Sec. VII B 1, that determine the correct diagonalization of the degenerate subspace in the matrix A . In the case of the cyclic state, the first three modes have displacements that correspond to rotations about three orthogonal axes. The final mode $z \propto \bar{u}_4$ corresponds to the optical excitation discussed above and its displacements are depicted in Fig. 3. This procedure simplifies the standard Bogoliubov method³⁹ considerably; we extract the eigenmodes solely from the A matrix, which, as we show next, can be obtained from symmetry considerations.

B. Spin-three hexagonal state

Let us now describe how to obtain the normal modes of a spinor condensate by using symmetry arguments alone in a more complicated setting. Once having the eigenmodes, however, we must note that to obtain the energetics and dispersions of these modes, analysis of the microscopic Hamiltonian is still required. Our analysis uses group theoretical arguments similar to those used to determine the vibrational modes of polyatomic molecules.^{35,36} We illustrate the method through the nontrivial example of the spin-three state having the symmetry of the hexagon, which is a candidate for the ground state of ^{52}Cr condensates.^{30,31}

The hexagon belongs to the point-symmetry group D_{6h} whose character table is given in Table I. In this table we use the standardized notation for the symmetry operators and irreducible representations.³⁵ To every spin node, we attach two displacement vectors parameterized by the real and imaginary parts of the z_i 's introduced previously. Such displacement vectors are always parallel to the surface of the sphere. We can construct matrices M_i which describe how this set of $2 \cdot 2F = 4F$ vectors transform under each of the symmetry operations. It is easy to then see that this set of

TABLE I. The character table of the group D_{6h} using the notation of Ref. 35. The last row gives the characters of the reducible representation Γ constructed from transforming the displacement vectors of the hexagon (see text).

D_{6h}	E	$2C_6$	$2C_3$	C_2	$3C'_2$	$3C''_2$	i	$2S_3$	$2S_6$	σ_h	$3\sigma_d$	$3\sigma_v$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1
A_{2g}	1	1	1	1	-1	-1	1	1	1	1	-1	-1
B_{1g}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
B_{2g}	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1
E_{1g}	2	1	-1	-2	0	0	2	1	-1	-2	0	0
E_{2g}	2	-1	-1	2	0	0	2	-1	-1	2	0	0
A_{1u}	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
A_{2u}	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
B_{1u}	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1
B_{2u}	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1
E_{1u}	2	1	-1	-2	0	0	-2	-1	1	2	0	0
E_{2u}	2	-1	-1	2	0	0	-2	1	1	-2	0	0
Γ	12	0	0	0	-4	0	0	0	0	0	0	0

matrices $\Gamma \equiv \{M_i\}$ form a (reducible) representation of the symmetry group. While these large $4F \times 4F$ matrices are cumbersome to write down, their characters (traces) can be obtained by inspection. For instance, only spin nodes which are mapped to themselves by a particular symmetry operation will contribute to the character of the matrix describing this symmetry operation. The last row of Table. I gives the characters of each of the matrices M_i forming Γ .

One can then invert the character matrix given in Table. I to see how Γ can be decomposed into combinations of irreducible representations. The result is

$$\Gamma = A_{2g} + B_{2g} + E_{1g} + E_{2g} + A_{2u} + B_{2u} + E_{1u} + E_{2u}. \tag{143}$$

In the typical notation³⁵ A 's and B 's denote one-dimensional irreducible representations while E 's denote two-dimensional irreducible representations. The normal modes form the basis of each of these irreducible representations.³⁵ For two-dimensional irreducible representations, there is some ambiguity in picking the two basis functions. For simplicity, we picked the particular displacements which are all in plane or all out of plane to form such bases. The $4F$ modes corresponding to each of these representations is given in Fig. 4. As usual, the modes corresponding to two-dimensional representations are degenerate.

Once we know the irreducible representations involved, we follow standard group theory, and construct projection operators for the modes in these irreducible representations. A general displacement of the spin nodes Q can be decomposed into a superposition of modes forming bases for each irreducible representation as

$$Q = \sum_i \mathcal{P}(\Gamma_i)Q, \tag{144}$$

where the operator $\mathcal{P}(\Gamma_i)$ projects into the irreducible representation Γ_i . Such projection operators can be written explicitly as

$$\mathcal{P}(\Gamma_i) = \frac{\ell_i}{h} \sum_g \chi^{(\Gamma_i)}(g)D(g). \tag{145}$$

Here, $\chi^{(\Gamma_i)}(g)$ is the character for the irreducible representation Γ_i corresponding to group element g , ℓ_i is the dimension of the i th irreducible representation, and h are the number of elements in the symmetry group; $D(g)$ is the representation of group element g in the spin-nodes displacement basis. For the hexagonal state of the spin-three condensate, this projection confirms the eigenmodes depicted in Fig. 4.

As mentioned above, unlike molecular normal modes where the atoms oscillate linearly about the equilibrium positions, the spin nodes will rotate along ellipses about the

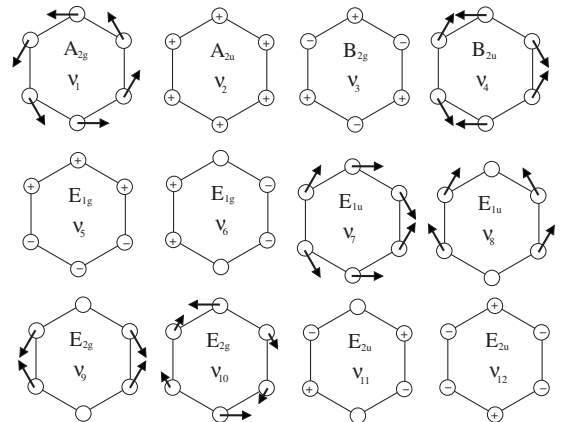


FIG. 4. Normal modes for the hexagonal configuration of the spin-three condensate. Vectors moving into and out of the plane are denoted with “-” and “+,” respectively. By multiplying the set of parameters $\{z_i\}$ corresponding to these displacements by a factor of i , one can identify $v_1 = v_2$, $v_3 = v_4$, $v_5 = v_7$, $v_6 = v_8$, $v_9 = v_{11}$, and $v_{10} = v_{12}$. The modes $v_1 = v_2$, $v_5 = v_7$, and $v_6 = v_8$ correspond to Goldstone excitations due to the broken-spin symmetry while all other modes are optical and gapped. The mode $v_{10} = v_{12}$ has a set of displacement vectors with lengths differing by a factor of 2.

equilibrium configuration. This allows us to cut the number of modes given Fig. 4 in half. Specifically, by multiplying the displacements $\{z_i\}$ by the phase factor of i , we identify $\nu_1=\nu_2$, $\nu_3=\nu_4$, $\nu_5=\nu_7$, $\nu_6=\nu_8$, $\nu_9=\nu_{11}$, and $\nu_{10}=\nu_{12}$. Because of rotational invariance, the aspect ratio of the ellipses for the three spin rotational Goldstone modes $\nu_1=\nu_2$, $\nu_5=\nu_7$, and $\nu_6=\nu_8$ will be zero. Finally, we identify the three remaining modes $\nu_3=\nu_4$, $\nu_9=\nu_{11}$, and $\nu_{10}=\nu_{12}$ with gapped optical modes of the hexagonal spin-three condensate.

Thus, for this spin-three problem, by symmetry arguments alone we have identified the $2F=6$ spin modes (three of which are Goldstone modes). These modes along with density mode give the complete spectrum of normal modes for the spin-three hexagonal condensate.

C. Connection to atomic-orbital theory and spherical harmonics

The treatment above makes the construction of low-energy eigenmodes of spinor-condensates geometrically intuitive and illustrates how to directly use the machinery of group theory. In addition, however, it is possible to make use of the close relationship of the symmetry group and the underlying full $SO(3)$ rotational symmetry (such a connection was explored in the context of equilibrium spinor condensates in Ref. 33). Once this connection is made, we will be able to simply map the already well-developed theory of crystal-field splittings of atomic orbitals to the problem of eigenmodes of spinor condensates.

The connection between the z representation of small oscillations as in Sec. VIII A and spherical harmonics can be deduced from the transformation rules of the vector z_i under the relevant point group. On the one hand, a symmetry operator in the z representation, $D_{ij}(g)$, will permute the entries z_i , as the symmetry operation g would the spin nodes. On the other hand, each z_i is a two-dimensional vector written in terms of a complex number with respect to a particular basis pair, $\bar{\mathbf{e}}_x, \bar{\mathbf{e}}_y$, which are functions of the location of the spin node \mathbf{n} on the unit sphere. Therefore, the operator $D_{ij}(g)$ also contains phase factors, $e^{i\lambda_{ij}(g)}$, which serve to rotate the basis vectors. So, in general, the structure of symmetry operators in the z basis is

$$D_{ij}(g) = A_{ij}^{(2F)}(g) e^{i\lambda_{ij}(g)}, \quad (146)$$

where $A_{ij}^{(2F)}(g)$ is an element of the $2F$ permutation group corresponding to a rotational symmetry of the spin nodes.

By exploiting the above transformation structure, we can systematically construct bases of the symmetry group over \mathbf{C}^{2F} from the bases of rotational symmetry, namely, spherical harmonics, $Y_{lm}(\theta, \phi)$. Let us mark the polar coordinates of the spin node \mathbf{n}_i as θ_i, ϕ_i ; from this set of coordinates, we can produce a $2F$ -dimensional complex vector

$$\{Y_{lm}(\theta_i, \phi_i)\}_{i=1}^{2F}.$$

It is easy to see that if we apply a rotational-symmetry operator g of the spinor condensate on this vector, we have

$$\sum_{m'} R_{mm'}^{(l)}(g) Y_{lm'}(\theta_i, \phi_i) = \sum_j A_{ij}^{(2F)}(g) Y_{lm}(\theta_j, \phi_j), \quad (147)$$

where $A_{ij}^{(2F)}(g)$ is the permutation operator from Eq. (146). The right-hand side of this equation indicates the rearrangement of the spin nodes due to the symmetry operator. On the other hand, the left-hand side comes from our knowledge of the transformation rules for spherical harmonics, under rotations: namely, l , the total angular momentum is invariant, and the different azimuthal angular-momentum components mix under the transformation.

To connect the spherical harmonics with the z representation, we need to construct a vector that will also transform with the phase $e^{i\lambda_{ij}(g)}$. This requires that in addition to evaluating the spherical harmonics at the points (θ_i, ϕ_i) , we need to account for the phase factor when constructing the derived bases in the z representation. This can be achieved by the following notion: instead of looking at the value of $Y_{lm}(\theta, \phi)$, let us look at its derivative, which in our gauge convention can be written as

$$\frac{\partial Y_{lm}(\theta, \phi)}{\partial z_i^*} = \left(\frac{\partial}{\partial \theta} + i \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) Y_{lm}(\theta, \phi). \quad (148)$$

The denominator of the partial derivative ∂z_i^* can be thought of as a small deviation, δz_i^* , from the mean-field spin node; it obeys the complex conjugate of the transformation rule in Eq. (146), so its inverse transforms in the correct way, using the phase $e^{i\lambda_{ij}(g)}$. Therefore, we finally have the connection between the symmetry of the spinor condensate and the representations of $SO(3)$

$$\sum_{m'} R_{mm'}^{(l)}(g) \frac{\partial Y_{lm'}(\theta_i, \phi_i)}{\partial z_i^*} = \sum_j D_{ij}(g) \frac{\partial Y_{lm}(\theta_j, \phi_j)}{\partial z_j^*}. \quad (149)$$

What we achieved by making this connection is a way of constructing for each $l > 1$ ($l=0$ gives identically zero) partially reduced (albeit still reducible) representations of the symmetry group at hand in terms of the z parametrization of small deviations from equilibrium. Let us denote the vectors we construct from Y_{lm} as

$$\mathbf{u}_{l,m} = \left\{ \frac{\partial Y_{lm}(\theta_i, \phi_i)}{\partial z_i^*} \right\}_{i=1}^{2F}. \quad (150)$$

These vectors are the simplest building blocks for the vibrational and rotational eigenvectors.

As an example, consider the $l=1$ states (p states) obtained for the cyclic state of spin-two condensates. With the orientation for the cyclic state given in Sec. VIII A, we obtain for $l=1$ and $m=0$

$$\mathbf{u}_{1,0} \propto \{\bar{\mathbf{e}}_{i+} \cdot \hat{x}_3\}_{i=1}^4 \propto (1, 1, 1, 1) \quad (151)$$

which is the displacement vector for $x_3=z$ axis rotation, as in Eq. (135). For $m=\pm 1$, as in atomic-orbital physics, it is useful to construct the p_x and p_y combinations, which are $p_{x,y} \propto Y_{1,1} \mp Y_{1,-1}$. For p_x we obtain

$$\mathbf{u}_{1,1} - \mathbf{u}_{1,-1} \propto \{\bar{\mathbf{e}}_{i+} \cdot \hat{x}_1\}_{i=1}^4 \propto \{1, -1, -1, 1\} \quad (152)$$

which is the rotation about the $x_1=x$ axis (up to an overall complex coefficient). In the same fashion we find that the p_y combination is

$$\mathbf{z}_{1,1} + \mathbf{z}_{1,-1} \propto \{\bar{\mathbf{e}}_{i+} \cdot \hat{x}_2\}_{i=1}^4 \propto \{1, -1, 1, -1\} \quad (153)$$

which corresponds to rotation about the $x_2=y$ axis. To obtain the last mode, which is the optical vibration mode shown in Fig. 3, all we need is to find the vector of \mathbf{u} which is orthogonal to the above three.

The above analogy with atomic p orbitals is not accidental. Since we mapped vibrational modes to spherical harmonics, we also mapped the z representation of spinor-condensate fluctuations to the lm representation of atomic orbitals. In atomic-orbital theory, we know that in the absence of rotational symmetry breaking all m states within the same l is degenerate. But in the presence of a crystal field, this degeneracy is lifted. The effect of crystal fields on angular-momentum multiplets is very well documented (see, e.g., Ref. 35); we can now use this resource to directly find the eigenmodes of the spinor condensates.

Let us demonstrate this principal again using the cyclic state. We have already shown that the $l=1$ vibration modes correspond to rotations. Let us now consider the $l=2$ states. Under the effect of a tetrahedral crystal field the electronic states split as

$$5d \rightarrow \begin{cases} d_{xy}, d_{xz}, d_{yz} \\ d_{z^2}, d_{x^2-y^2} \end{cases} \quad (154)$$

Now we can map back these atomic states to spinor-condensate oscillation modes. Starting with $d_{xy} \propto Y_{2,2} - Y_{2,-2}$ we find

$$\mathbf{u}_{2,2} - \mathbf{u}_{2,-2} \propto \{1, 1, 1, 1\} \quad (155)$$

which corresponds to uniform rotation about the z axis. Similarly $d_{xz} \propto Y_{2,1} - Y_{2,-1}$ and $d_{yz} \propto Y_{2,1} + Y_{2,-1}$ correspond to rotations about the y and x axis, respectively. The two remaining orbitals are $d_{z^2} \propto Y_{2,0}$ and $d_{x^2-y^2} \propto Y_{2,2} + Y_{2,-2}$. Since there are only four independent vectors, \mathbf{z} , d_{z^2} , and $d_{x^2-y^2}$ translate to the same z_{lm} vector

$$\mathbf{u}_{2,0} \propto \mathbf{u}_{2,2} + \mathbf{u}_{2,-2} \propto \{1, 1, -1, -1\} \quad (156)$$

which is exactly the optical mode shown in Fig. 3.

IX. CONCLUSIONS

One of the most striking and surprising features of spinor condensates is the hidden symmetry of their mean-field ground states. In this work, based on a spin-node description, we have strived to bring this symmetry to the forefront, and to make it into a tool in the study of the dynamics of these fascinating systems.

In the first part of this work, we derived the hydrodynamic equations of motion for condensates of general spin, demonstrated their use in the computation of the skyrmion configuration of a ferromagnetic spin-half gas, and generalized the Mermin-Ho relation to spin-one condensates.

In the second part of this work, we concentrated on small oscillations of the spinor condensate in the vicinity of the mean-field ground state. It is there that the hidden point-group symmetry becomes most apparent and accessible. Using the spin-node formalism and the parametrization of the spin nodes in terms of a stereographic projection, we reduced the problem of finding the $2F$ spin-wave eigenmodes to a simple question of decomposing a representation of the appropriate point symmetry group to its irreducible representations. We also provided a simple recipe that allows the direct extraction of the condensate's spin-wave eigenmodes using the derivatives of the spherical harmonics, coupled with the knowledge of atomic-orbital degeneracy lifting under a crystal field.

More than any specific result, this paper derives a new formalism to address high-spin many-body systems. It is our impression that, by far, we have not yet explored all possible applications of this formalism. A simple example is the calculation of the spin-wave eigenmodes and energies of a spinor condensate which is locally at its ground state but with its spin-nodes structure rotated as a function of space. This can be done by combining the linearization of Sec. VII with the general hydrodynamic description derived in Sec. VI. Similarly, our method of expanding about a mean-field ground state in terms of the z variables could be readily applied to computing the leading instabilities in quantum-quench experiments (as in, for instance, Ref. 17, where spin-one quantum-quench experiments were performed). The linearized Lagrangian derived in Sec. VII C applies near any extremum of the spin-interaction energy, V_{int} , even an unstable one. This can then be used to investigate the dynamics for short-time scales after a quantum quench.

Another possible direction focuses on the form of the spin-interaction energy $V_{\text{int}}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \dots, \hat{\mathbf{n}}_{2F})$, first defined in Eq. (10). In terms of the spin nodes, the spin-interaction energy must be a permutation symmetric function of the spin nodes. But the number of permutation symmetric scalars constructed of the spin nodes $\hat{\mathbf{n}}_i$ is limited. All such scalars must be constructed from tensors of the form

$$\mathcal{M}_{\alpha_1 \alpha_2 \dots \alpha_n} = \sum_{i=1}^{2F} n_{i, \alpha_1} n_{i, \alpha_2} \dots n_{i, \alpha_n}, \quad (157)$$

where $\alpha_k = x, y, z$ is the space direction. Examples are

$$\sum_{i,j=1}^{2F} \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = \sum_{i,j=1}^{2F} n_{i,\alpha} n_{j,\alpha} \sum_{i,j=1}^{2F} n_{i,\alpha} n_{i,\beta} \cdot n_{j,\alpha} n_{j,\beta} \quad (158)$$

and so forth. This structure of the spin interaction may be used to construct generic phenomenological theories for spinor condensates and other high-spin many-body systems, along the lines of the construction of Landau free energy.

The most interesting applications of the spin-node formalism may arise when considering noncondensed spinor systems. Lattice insulators, both fermionic and bosonic, could also be parametrized using spin nodes, and should exhibit magnetic mean-field states with hidden point-group symmetry as well. Similarly, we intend to consider spinor *Fermi liquids* using this formalism; such systems may have inter-

esting magnetic instabilities into states with the same hidden symmetries as those arising in spinor condensates.

In the challenging field of many-body quantum systems, often a new technical perspective on a problem may simplify it dramatically. In this paper we developed a formalism that seeks to do exactly that to the dynamics of spinor condensates—a topic of much current experimental as well as theoretical interest. Our analysis provides an economical representation, which allows for a direct, general, and easy calculation of many dynamic collective properties of spinor condensates. In addition, we hope that the developments presented here could be used in other challenging problems involving interacting quantum systems with high spin.

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APPENDIX A: NOTATION

In this Appendix, for convenience, we collect in one place the notation used for the representation of the spinors. Normalized spinors of arbitrary spin are denoted by $|\chi\rangle$. Non-normalized symmetric combinations of spin-half states are denoted as (with bold fonts)

$$|\mathbf{\Omega}\rangle = |\Omega_1\Omega_2\cdots\Omega_{2F}\rangle. \quad (\text{A1})$$

This is then related to $|\chi\rangle$ by

$$|\chi\rangle = \frac{|\mathbf{\Omega}\rangle}{\sqrt{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle}}. \quad (\text{A2})$$

Coherent spin states occur when all of the spin-half constituent spins point in the same direction. We denote these by

$$|(\Omega_i)^{2F}\rangle = |\Omega_i\Omega_i\cdots\Omega_i\rangle. \quad (\text{A3})$$

We next define the spin state corresponding to $|\mathbf{\Omega}\rangle$ with its i th component time reversed. We denote these by

$$|T_i\mathbf{\Omega}\rangle = |\Omega_1\Omega_2\cdots\Omega_i'\cdots\Omega_{2F}\rangle. \quad (\text{A4})$$

Finally, we define the projection operator \mathcal{P} to be

$$\mathcal{P} = 1 - |\chi\rangle\langle\chi|. \quad (\text{A5})$$

APPENDIX B: THE SPINOR BASIS $\{|T_i\mathbf{\Omega}\rangle\}$ AND THE MATRIX A

In this Appendix, we will develop derive identities used for the projection operator $\mathcal{P} = 1 - |\chi\rangle\langle\chi|$ and the symmetry matrix A . Consider a particular spinor

$$|\mathbf{\Omega}\rangle = |\Omega_1\Omega_2\cdots\Omega_{2F}\rangle, \quad (\text{B1})$$

where none of the spin nodes are degenerate. Then from this we can construct a set of $2F$ states where one of the elements of $|\mathbf{\Omega}\rangle$ is time reversed $\{|T_i\mathbf{\Omega}\rangle\}$. Furthermore, we construct the set of $2F$ coherent states which are orthogonal to $|\mathbf{\Omega}\rangle$ which are $\{(\Omega_i')^{2F}\}$. We note that these two sets of states satisfy reciprocal relations

$$\langle T_i\mathbf{\Omega} | (\Omega_i')^{2F} \rangle = \lambda_i \delta_{ij}, \quad (\text{B2})$$

where

$$\lambda_i = (2F)! \prod_{j \neq i} \langle \Omega_j | \Omega_i' \rangle. \quad (\text{B3})$$

This relation leads to a useful identity for the projection operator

$$\mathcal{P} = \sum_i \frac{|(\Omega_i')^{2F}\rangle\langle T_i\mathbf{\Omega}|}{\lambda_i} \mathcal{P}. \quad (\text{B4})$$

This relation can be immediately proved by expanding any state acting on the right in a basis of states $\{(\Omega_i')^{2F}\}$, and any state acting on the left in a basis of states $\{|T_i\mathbf{\Omega}\rangle\}$ (both which, in addition to the state $|\mathbf{\Omega}\rangle$, form a complete basis of spinor states when the spin nodes are nondegenerate).

Using these states, we will now proceed to derive an expression for the inverse of the matrix $A_{ij} = \langle T_i\mathbf{\Omega} | \mathcal{P} | T_j\mathbf{\Omega} \rangle$ which exists when none of the spin nodes \mathbf{n}_i are degenerate. We define B to be the matrix of the overlap of time-reversed coherent states (which will be shown to be the inverse of A)

$$B_{ij} = \frac{\langle (\Omega_i')^{2F} | (\Omega_j')^{2F} \rangle}{\lambda_i^* \lambda_j}. \quad (\text{B5})$$

Consider the product of these matrices

$$\sum_j B_{ij} A_{jk} = \sum_j \frac{\langle (\Omega_i')^{2F} | (\Omega_j')^{2F} \rangle}{\lambda_i^* \lambda_j} \langle T_j\mathbf{\Omega} | \mathcal{P} | T_k\mathbf{\Omega} \rangle. \quad (\text{B6})$$

We can then use the identity in Eq. (B4) to collapse the sum over j . This leads to

$$\sum_j B_{ij} A_{jk} = \frac{\langle (\Omega_i')^{2F} | T_k\mathbf{\Omega} \rangle}{\lambda_i^*} = \delta_{ik} \quad (\text{B7})$$

and the proof is complete.

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